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ABSTRACT

This is part two of a two-part manual for teachers using SMSG high school text materials. The manual contains a chapter-by-chapter commentary on the text, solutions to the problems in the text, and a collection of essays on topics related to material in the text. Chapter topics include: (1) perpendicular lines and planes in space; (2) parallel lines in a plane; (3) parallel lines in space; (4) areas of polygonal regions; (5) similarity; (6) circles and spheres; (7) loci and constructions; (8) the area of a circle and related topics; and (9) plane coordinate geometry. (MP)

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**SCHOOL
MATHEMATICS
STUDY GROUP**

MATHEMATICS FOR HIGH SCHOOL

GEOMETRY (Part 2)

COMMENTARY FOR TEACHERS

(preliminary edition)

2

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THE OHIO STATE UNIVERSITY
CENTER FOR SCIENCE AND MATHEMATICS EDUCATION
248 Arps Hall - 1945 North High Street
Columbus 10, Ohio

MATHEMATICS FOR HIGH SCHOOL

GEOMETRY (Part 2)

COMMENTARY FOR TEACHERS

(preliminary edition)

Prepared under the supervision of the Panel on Sample Textbooks of the School Mathematics Study Group:

Frank B. Allen	Lyons Township High School
Edwin C. Douglas	Taft School
Donald E. Richmond	Williams College
Charles E. Rickart	Yale University
Henry Swain	New Trier Township High School
Robert J. Walker	Cornell University

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GEOMETRY

Teacher's Manual

Part I

Teacher's Commentary

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Chapter 8

223 In the discussion preceding Theorem 8-1, we introduce the idea of perpendicularity of a line and a plane and are trying to convince ourselves that if L is perpendicular to two lines passing through P, then L will be perpendicular to E.

224 Note that in Theorem 8-1 we are working in three-space, and that the points P and Q do not have to be co-planar with A, B, and X on line L. The modification of the proof when B and X are on opposite sides of A merely requires that the roles of A and B be interchanged, and then the proof remains the same.

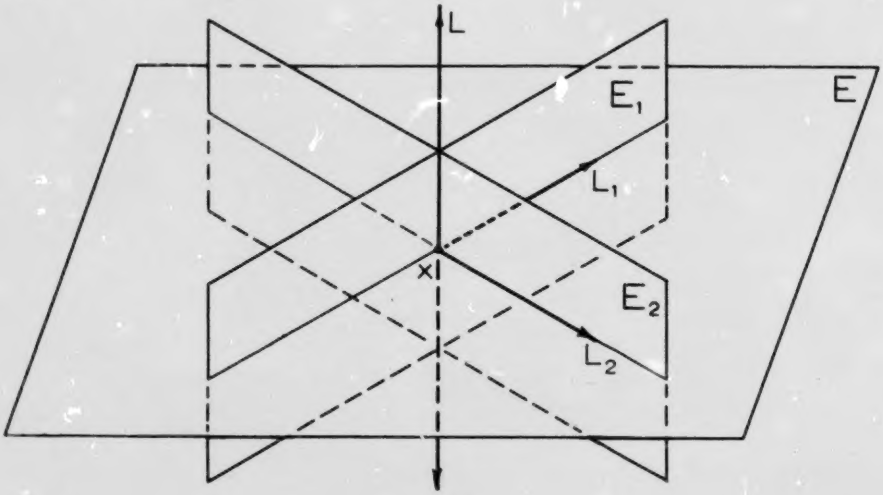
224 and 225 To prove that X is equidistant from P and Q in Theorem 8-2, we show that it lies on a line, two of whose points are equidistant from P and Q. We do this by finding two points, Y and Z in plane E, such that X, Y and Z are collinear and Y and Z are each equidistant from P and Q, and apply Theorem 8-1 to get X equidistant from P and Q. Note carefully how the points Y and Z were obtained.

In the first part of this proof we are establishing that B_1 is equidistant from P and Q. This is done by proving $\triangle PAB_1 \cong \triangle QAB_1$. The lines necessary to complete the triangles are not drawn in the text since this is the only use to which they will be put, and any additional lines might only tend to confuse the diagram and the student. If it would help your students to visualize this, draw the lines on the board, or better still have them make a model using string for the lines. Once we have established that three points (B_1, B_2, and A) in E are non-collinear and equidistant from P and Q, then it follows from Theorem 8-2 that every other point in this plane is equidistant from P and Q. Hence we can prove $\triangle PAB_3 \cong \triangle QAB_3$, and by corresponding parts establish that $\angle PAB_3$ and $\angle QAB_3$ are right angles.

226 Note that Theorem 8-4 refers to "the perpendicular bisector" only for convenience of expression. The restatement clearly shows that we are not assuming that there is a unique perpendicular bisecting plane of a segment. This is settled by Theorem 8-11.

The same observation applies to Theorem 8-5.

229 Proof of Theorem 8-7: Through a given point on a given line there is a plane perpendicular to the given line.



Proof: Given line L and point X on L: Let E_1 be a plane containing L. By Theorem 5-5 there is in E_1 a line $L_1 \perp L$ at X. Similarly let E_2 be another plane containing L. Then there is in E_2 a line $L_2 \perp L$ at X. Let E be the plane determined by L_1 and L_2. By Theorem 8-3, $L \perp E$ at X or $E \perp L$ at X.

We have tacitly assumed that there are at least two planes containing L. This can be proved from our Postulatesas follows: By Postulate 1.c there exist four non-coplanar points A, B, C, D. If three of these, say A, B, C, are on L then A, B, C lie in a plane by Postulate 3. Then L lies in this plane by Postulate 4. Suppose A, B, C are not all on L, say A is not on L. Then A and L determine a plane by Theorem 3-3. Thus in any case L is contained in a plane E_1. Since A, B, C, D are not coplanar, one of them, say B, is not in E_1. Then L and B determine a plane E_2. Clearly $E_2 \neq E_1$, since B is in E_2 but not in E_1.

Using the Ruler Postulate we can show that there are infinitely many points on a line, and that there are infinitely many planes containing a given line.

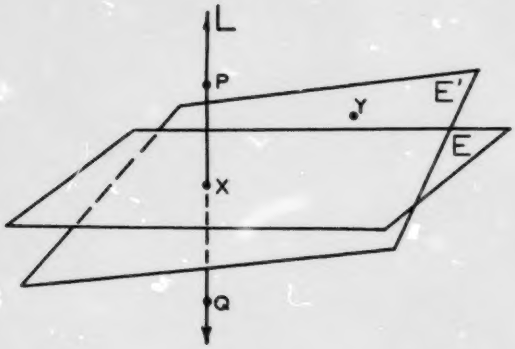
229 Proof of Theorem 8-8: Through a given point in a given plane there is a line perpendicular to the plane. Since $L \perp E_1$, then $L \perp L_2$, by the definition of a line perpendicular to a plane, and $L_2 \perp L_1$ by construction. We have L_2 perpendicular to two intersecting lines in E (L_1 and L) at their point of intersection.

230 In the proofs of Theorems 8-8 and 8-9, there is a common theme. Intuitively think of E as "horizontal." Then in different ways, as indicated in the proofs of these theorems, we construct E_1 as a "vertical" plane containing P. Then we draw a perpendicular in this "vertical" plane from P to the line of intersection of the two planes.

Hence by Theorem 8-3, $L_2 \perp E$.

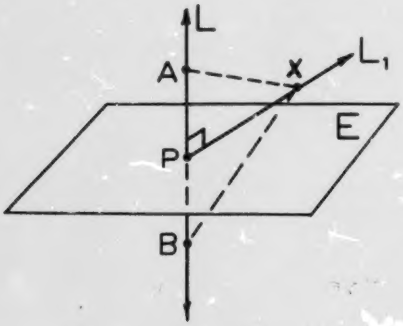
In the proof of Theorem 8-9, if L_1 passes through P, then L_1 is the perpendicular we are looking for. Hence we assume L_1 does not pass through P.

232 Proof of Theorem 8-11: Through each point of a line there is only one plane perpendicular to the line.



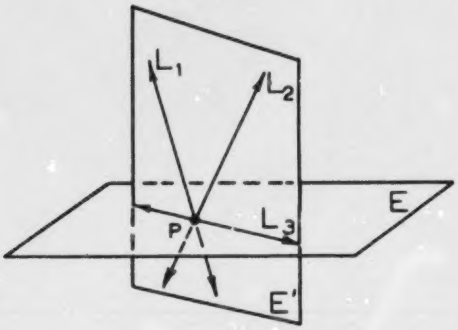
Suppose two planes, E and E' are perpendicular to line L at a point X. Let P and Q be points of L such that X is between P and Q, and PX = QX. By Theorem 8-4, every point Y of plane E' is equidistant from P and Q. Then by Theorem 8-5, Y must be in E. That is, all points of E' are in E. Hence E and E' must coincide and there is only one plane perpendicular to L at X.

233 Proof of Corollary 8-11-1: If plane E and line L are perpendicular, intersecting in a point P, then E contains every line through P, perpendicular to L.



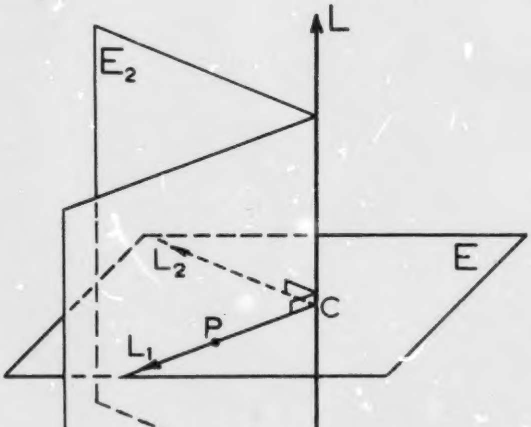
Let L_1 be perpendicular to L at P. Let A and B be points of L such that P is between A and B and AP = PB. Let X be any point on L_1 different from P. Then $\triangle APX \cong \triangle BPX$ by S.A.S. Postulate. By corresponding parts AX = BX, hence X is equidistant from A and B. Then by Theorem 8-5 X is in plane E; therefore L_1 lies in E.

233 Proof of Theorem 8-12: Through a given point in a plane there is only one line perpendicular to the plane.



Suppose that L_1 and L_2 are two different lines, both containing P and both perpendicular to E. Let E' be the plane that contains L_1 and L_2. Then E' intersects E in a line L_3. Therefore E' contains two lines through P, both perpendicular to L_3. This contradicts Theorem 5-5, hence there is only one line perpendicular to plane E at P.

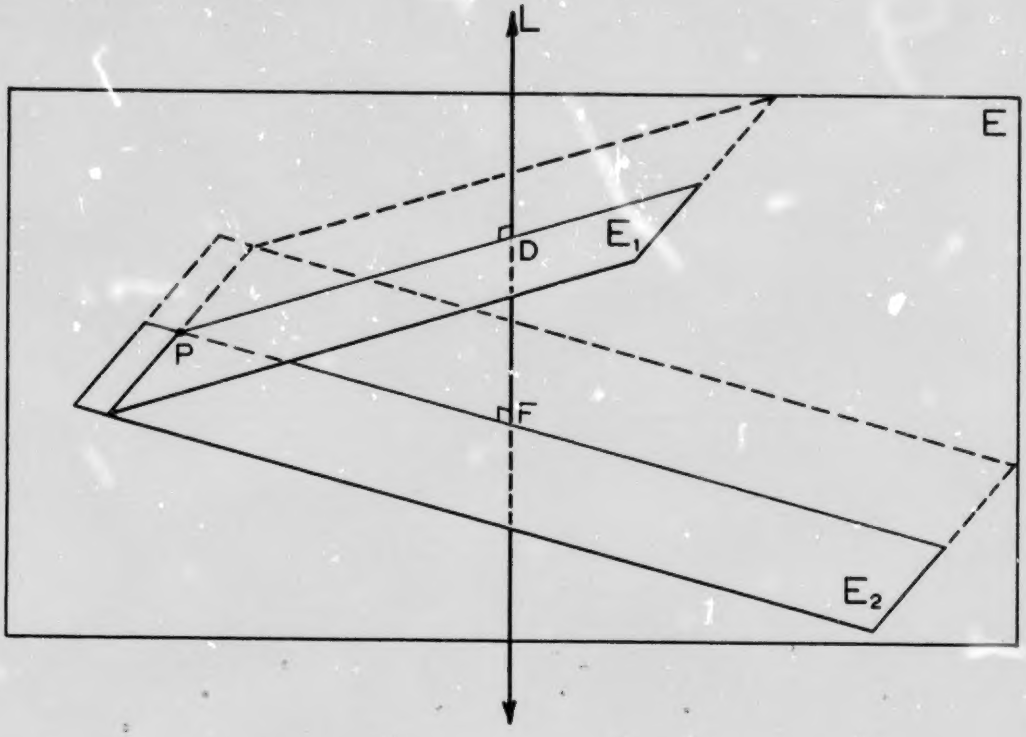
233 Proof of Theorem 8-13: Through a given external point there is one and only one plane perpendicular to a given line.



Let L be the given line and P the external point. P and L determine a plane E_1. In E_1, draw the perpendicular from P to L, meeting L at C. Call this line L_1. Let D be any point not in plane E_1. D and L determine a plane E_2. In E_2 draw the line perpendicular to L at C. Call this L_2. Since L is perpendicular to L_1 and L_2 at C, then L is perpendicular to the plane E, determined by L_1 and L_2, and plane E contains P because P is on L_1. We see that plane E is perpendicular to L and contains the external point P.

Hence, we have proved that there is a plane through P perpendicular to L.

Now we must show that this is the only plane which satisfies this condition.



Let us assume that there are 2 planes through P perpendicular to L, E_1 and E_2, intersecting L at points D and F respectively. Hence, in E_1 $\overleftrightarrow{PD} \perp L$ and in E_2 $\overleftrightarrow{PF} \perp L$, L and P determine a plane E. Hence in E we have two lines ($\overleftrightarrow{PD}$ and $\overleftrightarrow{PF}$) $\perp$ L, which contradicts the corollary to Theorem 7-1.

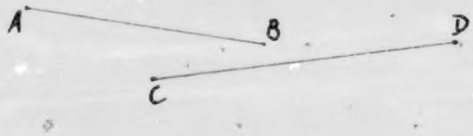
Chapter 9.

The discussion of parallel lines in a plane, though by no means difficult, encompasses probably the most significant property of Euclidean geometry, namely, the "Parallel Postulate", stated on page 253 . By means of introduction ask the students to tell what they mean by parallel lines. The answers will no doubt vary, and some will probably be incorrect. Most answers will probably be descriptions, rather than definitions. It is hoped that from a discussion of this sort the class will get the feeling that they are working with something that is intuitively very simple, but that at the same time the concept of parallelism is not one that can easily be "pinned down" by the student.

Remind the students that parallel lines do not meet. You will sometimes hear the expression: "parallel lines meet at infinity". This is just another way of saying that the lines do not meet. Mathematicians abhor exceptions, for example, two lines do not always determine a point in the Euclidean plane, and just as it is convenient to introduce complex numbers into algebra so that every quadratic equation has a root, so it is convenient to adjoin to the real points of the plane, certain "ideal" points so that we can say two lines always meet, but notice that such lines are no longer Euclidean lines.

To each Euclidean line we adjoin an ideal point to form a new kind of line, called a projective line, that is no longer a Euclidean line. This is done in such a way that the same ideal point is adjoined to each line of a family of parallel lines. If two Euclidean lines are parallel then their associated projective lines meet in an ideal point. If two Euclidean lines are not parallel they meet in a point P and their associated projective lines meet in the same point P. This avoids an exception, but all the properties of real points do not carry over automatically to ideal points. When we say two projective lines meet at an ideal point, it follows that their associated Euclidean lines do not meet at all. If we adjoin these ideal points to the set of real points in the Euclidean plane, we get a new "plane", which has different properties from the Euclidean plane, and which we may call a "projective plane" in the sense that "point", "line", and "plane" would satisfy the set of incidence postulates usually made for projective geometry. But this is not the geometry we are studying; in Euclidean Geometry we do have parallel lines, in Projective Geometry there are no parallel lines.

243 Point out to the students the definition of parallel lines gives two conditions that must be met by the lines, (1) they must lie in the same plane and (2) they must not intersect. Ask the student for an example of two lines that satisfy condition (2), they do not intersect, but fail to satisfy (1) and hence are not parallel. Skew lines in space is the example. Ask the students to criticize this definition: "two segments are parallel if (a) they lie in the same plane and (b) they do not intersect". Have them give you an example of two segments satisfying (a) and (b) but still are obviously not parallel. Here is a picture of an example that they might give you:

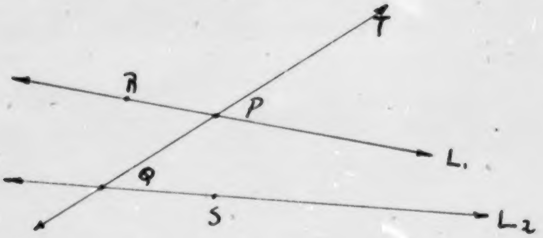


244 Here we again see an indirect proof using the method of contradiction. Two lines in a plane can be either parallel or they intersect. We assume they intersect, and find that this leads to a contradiction of previously established information, hence the remaining possibility must be true.

245 Theorem 9-1 gave us one method for constructing a line parallel to another line through an external point. The method was used in Theorem 9-2 to prove the existence of at least one parallel to a line from a point not on the line. Notice how the exterior angle theorem was used in the proof of Theorem 9-1. In the proof of Theorem 9.1 we used the exterior angle theorem because the same method of proof applies to Theorem 9.4.

The "one parallel" referred to in the discussion after Theorem 9-2 refers to the uniqueness of a parallel to a line through an external point. More will be said of this later.

246 Alternate interior angles may be formally defined as follows:



If two lines L_1 and L_2 are cut by transversal T in points P and Q respectively, and points R on L_1 and S on L_2 are on opposite sides of T then the two angles, ∠RPQ and ∠PQS, are called alternate interior angles.

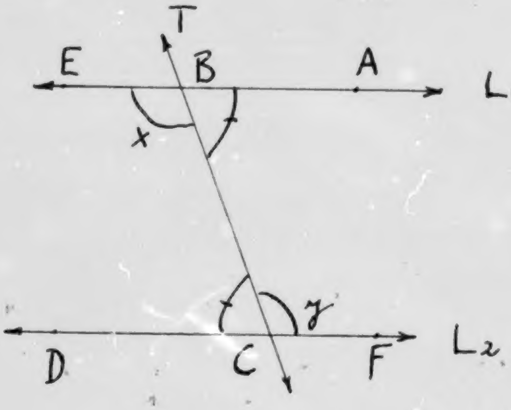
247 We now state and prove Theorem 9-3: Given a transversal to two lines, if one pair of alternate interior angles are congruent, then the other pair of alternate interior angles are also congruent.

Given: Lines L_1 and L_2 cut by transversal t such that

$\angle ABC \cong \angle BCD$

To Prove: $\angle x \cong \angle y$.

Since a point on a line separates the line into two opposite rays, we have $\overrightarrow{BA}$ and $\overrightarrow{BE}$; $\overrightarrow{CF}$ and $\overrightarrow{CD}$ as pairs of opposite rays. By Postulate 14 $\angle ABC$ and $\angle x$ are supplementary, as are $\angle BCD$ and $\angle y$. Since $\angle ABC \cong \angle BCD$, then $\angle x \cong \angle y$, because supplements of congruent angles are congruent.



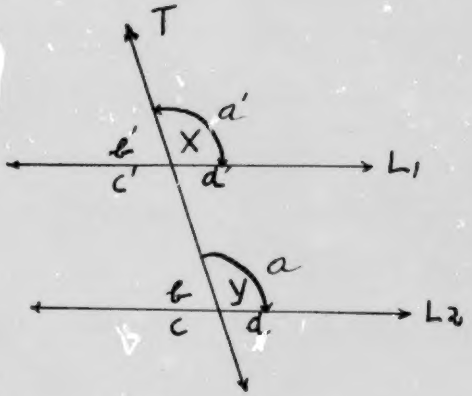
252 Statement and proof of Theorem 9-5: Given two lines and a transversal, if one pair of corresponding angles are congruent, then the other three pairs of corresponding angles have the same property.

Given: Lines L_1 and L_2 cut by transversal T such that a pair of corresponding angles, $\angle a$ and $\angle a'$, are congruent.

To Prove: $\angle b \cong \angle b'$, $\angle c \cong \angle c'$, $\angle d \cong \angle d'$.

Let us consider the case of one pair of these angles, and show that in the same manner the remaining two pairs can be proved congruent.

$\angle a \cong \angle a'$, Given.



Points X and Y on L_1 and L_2 separate these lines respectively into two opposite rays. From Postulate 14 we then have $\angle a'$ supplementary to $\angle b'$, and $\angle a$ supplementary to $\angle b$. Since supplements of congruent angles are congruent, $\angle b \cong \angle b'$.

Since X and Y separate the transversal T into pairs of opposite rays, again by using Postulate 14 and the property of supplements of congruent angles, we can establish the remaining pairs of corresponding angles congruent.

REMARK on the proof of Theorem 9-5: The students will devise many different proofs for this theorem. Here is one variation.

Given: Two lines L_1 and L_2 cut by transversal T such that a pair of corresponding angles, $\angle a$ and $\angle a'$, are congruent.

Let the angles be labeled as in the figure above.

To Prove: $\angle b \cong \angle b'$, $\angle c \cong \angle c'$, and $\angle d \cong \angle d'$.

Since vertical angles are congruent:

(1) $\angle a' \cong \angle c'$ (3) $\angle b' \cong \angle d'$

(2) $\angle a \cong \angle c$ (4) $\angle b \cong \angle d$

Since (5) $\angle a \cong \angle a'$ given

(6) $\angle a \cong \angle c'$ substituting from (5) into (1).

We now have a pair of alternate interior angles congruent, hence by Theorem 9-3 the other pair of alternate interior angles are congruent that is:

(7) $\angle b \cong \angle d'$

Therefore: (8) $\angle b \cong \angle b'$ substituting from (3) into (7)

(9) $\angle c \cong \angle c'$ " " " (2) into (6)

(10) $\angle d \cong \angle d'$ " " " (4) into (7).

The method of proof of Theorem 9-6 is merely to use the property of vertical angles to establish a pair of alternate interior angles congruent, and by Theorem 9-4, the lines are parallel.

252-253 The reason for the statement about proving the converse of Theorem 9.4 and 9.6 is that on the basis of our previous information we do not know if there is only one parallel to a line through an external point. And, in fact, on this basis, we cannot prove the uniqueness of a parallel through an external point, and thus we must assume Postulate 16 (The Parallel Postulate). Instead of assuming our Parallel Postulate, some geometers felt that it was just as valid to postulate that there were many or no parallels through an external point. After all, there is no theoretical reason why these latter statements are not just as valid as our Postulate 16. It is on the basis of our experience in the world about us that makes our postulate seem reasonable. Some type of parallel postulate must be assumed (see the earlier discussion concerning the concept of parallelism). From this point on, different Parallel Postulates result in the development of different geometries, called Non-Euclidean geometries. For further information on this subject you may wish to refer to the reading list at the end of this publication and to Chapter 1 of SMSG Studies in Mathematics Vol. II.

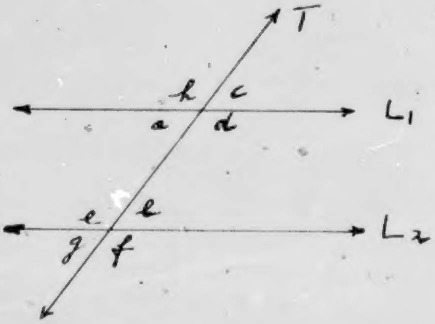
254 Now that we have the uniqueness of a parallel through an external point it is possible to prove the converse of Theorem 9.4. Note carefully in the proof in the text how the fact that this parallel is unique is used to establish the validity of Theorem 9.7. If Theorem 9.7 is assumed as our "Parallel Postulate" then it is possible to prove our Postulate 16. In doing so we would have proved the uniqueness of a parallel to a line through an external point parallel to a given line and proceed in the proof to a contradiction of the angle construction Postulate 12.

It is necessary that some type of parallel postulate be stated. From the discussion concerning Pg. 9.8 we see this is essential, inasmuch as there is a Non-Euclidean geometry which satisfies all of the postulates we have made thus far, but deny our parallel postulate.

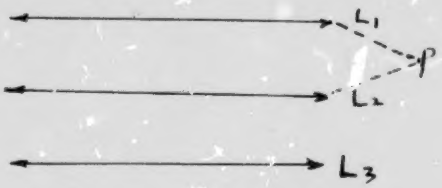
255 Proofs of Theorems 9-8 thru 9-11:

Theorem 9-8: If $L_1 \parallel L_2$, we then know by Theorem 9-7 that the alternate interior angles are congruent. By application of the property that vertical angles are congruent, we can establish the pairs of corresponding angles to be congruent.

Theorem 9-9: Given $L_1 \parallel L_2$. Then it follows from Theorem 9-7 that $\angle a \cong \angle b$, and since L_1 is a line, then $\angle a$ and $\angle d$ are supplementary. By substitution, then $\angle b$ and $\angle d$ are supplementary.

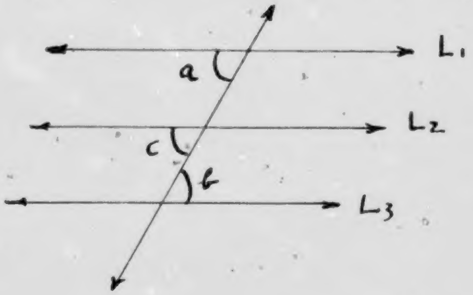


In a like manner $\angle e$ can be proved supplementary to $\angle a$.



Theorem 9-10: Given: L_1 and L_2 both $\parallel L_3$. To Prove: $L_1 \parallel L_2$. We will use the indirect method of proof and assume that L_1 is not parallel to L_2. If this is true, then these two lines will meet at some point P. This means that there are now two lines through P (L_1 and L_2) parallel to L_3. This contradicts the "Parallel Postulate", hence L_1 must be parallel to L_2. There is one restriction needed in the above proof. That is P cannot lie on L_3, since L_1 is parallel to L_3 and also $L_2 \parallel L_3$.

Remark on proof of Theorem 9-10: This theorem can be proved directly as follows:



Given: $L_1 \parallel L_3$, $L_2 \parallel L_3$.

To Prove: $L_1 \parallel L_2$.

Let T be a transversal intersecting L_1, L_2, and L_3 and consider the alternate interior angles formed as indicated in the figure.

Proof: $L_1 \parallel L_3$ hence (1) $\angle a \cong \angle b$ by Theorem 9-7.

$L_2 \parallel L_3$ " (2) $\angle c \cong \angle b$ " " 9-7.

(3) $\angle a \cong \angle c$ by substitution from (1) into 2.

Therefore since $\angle a$ and $\angle c$ are a pair of congruent corresponding angles, then by Theorem 9-6 $L_1 \parallel L_2$.

Theorem 9-11: By the properties of alternate interior angles or corresponding angles, it follows that if a transversal makes a right angle with one of two parallel lines, it must make right angles with the second parallel line.

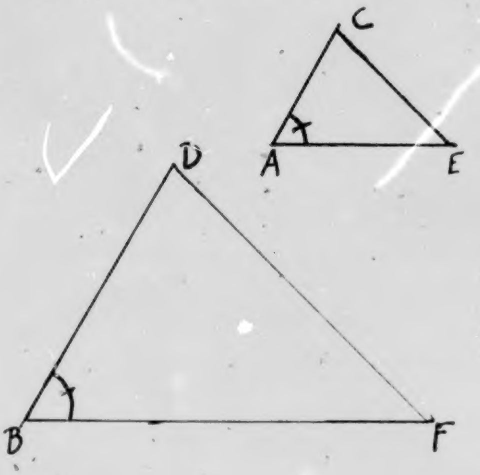
259 Theorem 9-12 is the first major consequence of our Parallel Postulate. The proof is directly related to the fact that there is but one line parallel to the base of the triangle through the opposite vertex. If there were more than one, or no parallels, the sum of the measures of the angles of a triangle would be less than 180 or greater than 180 as is the case in the Non-Euclidean Geometries. In hyperbolic geometry the sum of the measures of angles of a triangle is less than 180, but in spherical geometry the sum is greater than 180.

Proof of Corollary 9-12-1. Given a correspondence between two triangles. If two angles of the first triangle are congruent to the corresponding parts of the second, then the third angles are congruent.

Given: $\triangle ACE$ and $\triangle BDF$, such that $\angle A \cong \angle B$ and $\angle C \cong \angle D$.

To Prove: $\angle E \cong \angle F$.

We now know, from Theorem 9-12 that the sum of the measures of the angles of a triangle is 180. Given that the sums of the measures of two angles in each triangle are equal, then the differences between this sum and 180 in each case are equal.



If $m\angle E = m\angle F$ then $\angle E \cong \angle F$.

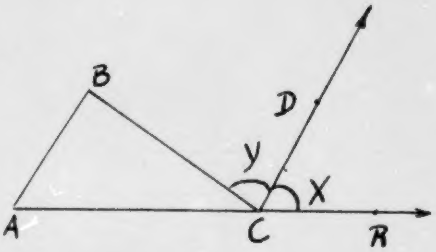
260 The proof of Corollary 9-12-2 is immediate since Corollary 9-12-1 tells us that the three corresponding angles are congruent and now in the two triangles $\triangle ABC$ and $\triangle DEF$ we have $\triangle ABC \cong \triangle DEF$

by A.S.A.

The proof of Corollary 9-12-3 is essentially the same as above. If the sum of the measures of the angles of a triangle is 180, and one angle has a measure of 90, then the sum of the measures of the remaining two angles must be 90. By definition, then, these angles are complementary.

Proof of Corollary 9-12-4.

Given: $\triangle ABC$ with R a point on $\overrightarrow{AC}$ such that C is between A and R.



To Prove: $m\angle BCR = m\angle A + m\angle B$.

1. Through C draw ray $\overrightarrow{CD} \parallel$ to $\overrightarrow{AB}$, as shown in the figure.	1. That a $\overrightarrow{CD}$ exists is established by Theorem 9-2.
2. $m\angle BCR = m\angle y + m\angle x$	2. Angle Addition Postulate.
3. $\angle B \cong \angle y$	3. Alternate interior angles.
4. $\angle A \cong \angle x$	4. Corresponding angles.
5. We can write statements 3 and 4 in their equivalent form: $m\angle B = m\angle y$, $m\angle A = m\angle x$.	
6. $m\angle BCR = m\angle A + m\angle B$	6. Substitution.

Here is an alternate proof for Corollary 9-12-4.

Using the above figure and the accompanying data

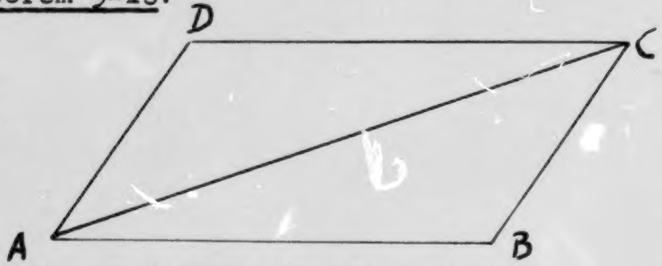
$m\angle BCR = 180 - m\angle BCA$ by the Supplement Postulate

$m\angle A + m\angle B = 180 - m\angle BCA$ by Theorem 9-12

Therefore $m\angle BCR = m\angle A + m\angle B$ by substitution.

The statement "ABCD is a ▱" is read, ABCD is a parallelogram. "▱ ABCD" is read, parallelogram ABCD.

267 Proof of Theorem 9-13:



Given: Parallelogram ABCD with diagonal AC.

To Prove: $\triangle ABC \cong \triangle CDA$

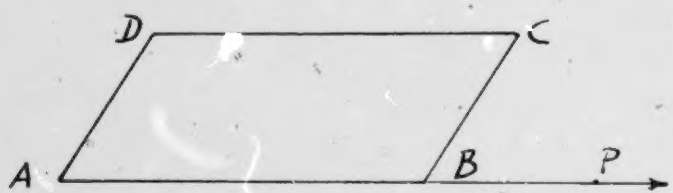
1. $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{CD}$	1. Definition of a parallelogram.
2. $\angle DCA \cong \angle CAB$ $\angle DAC \cong \angle ACB$	2. Alternate interior angles.
3. $\overline{AC} = \overline{CA}$	3. Identity.
4. $\triangle ABC \cong \triangle CDA$	4. A.S.A. Theorem.

The proof using diagonal $\overline{BD}$ is similar to this one given for diagonal $\overline{AC}$.

Theorem 9-14 is an immediate consequence of Theorem 9-13, since the triangles are congruent it follows that the corresponding sides are congruent.

For Theorem 9-15, if the triangles into which a diagonal divides a parallelogram are congruent, then the corresponding angles are congruent. In the figure above, $\overline{AC}$ divides the figure into $\triangle$ ABC and CDA which gives us $\angle D \cong \angle B$. By drawing diagonal $\overline{DB}$, we can establish, in the same manner, $\angle A \cong \angle C$.

Proof of Theorem 9-16.



Given: Parallelogram ABCD with P a point on $\overrightarrow{AB}$ such that B is between P and A.

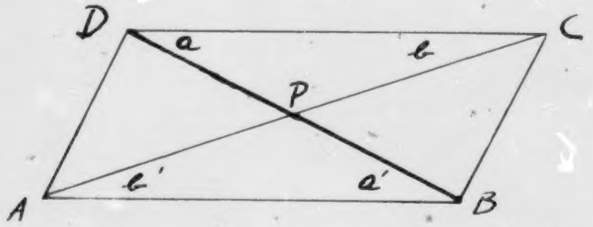
To Prove: $\angle A$ and $\angle ABC$ are supplementary.

1. $\angle A \cong \angle CBP$	1. Corresponding angles.
2. $\angle ABC$ and $\angle CBP$ are supplementary	2. Postulate 14.
3. $\angle A$ and $\angle ABC$ are supplementary	3. Substitution.

Since $\angle A \cong \angle C$, $\angle ABC$ is supplementary to $\angle C$, and since $\angle ABC \cong \angle D$, $\angle C$ is supplementary to $\angle D$, and $\angle D$ is supplementary to $\angle A$.

Remark on the proof of Theorem 9-16. Consider any two consecutive angles of a parallelogram as the interior angles on the same side of a transversal cutting two parallel lines. Then by Theorem 9-9 ("Given two parallel lines and a transversal, interior angles on the same side of the transversal are supplementary.") this theorem is proved.

268 Proof of Theorem 9-17:



Given: Parallelogram ABCD with diagonals $\overline{AC}$ and $\overline{BD}$. We assume from the figure that they intersect at P.

To Prove: $\overline{AC}$ and $\overline{BD}$ bisect each other.

1. $\angle a \cong \angle a'$ $\angle b \cong \angle b'$	1. Alternate interior angles.
2. $\overline{AB} \cong \overline{CD}$	2. Opposite sides of a ▱ are congruent.
3. $\triangle ABP \cong \triangle CDP$	3. A.S.A. Theorem.

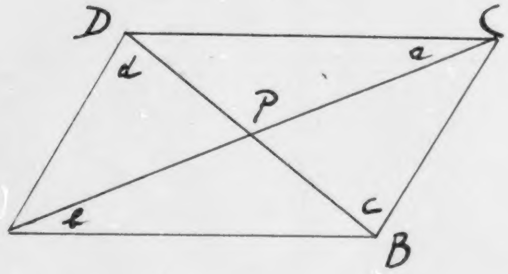
4. $\overline{AP} \cong \overline{CP}$ $\overline{DP} \cong \overline{BP}$	4. Corresponding parts.
5. Hence $\overline{AC}$ and $\overline{BD}$ bisect each other.	5. Statement 4 and Definition of bisection.

268

Proof of Theorem 9-18:

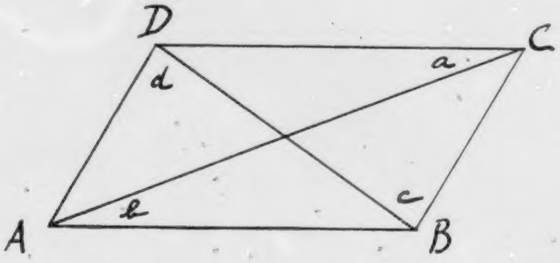
Given: Quadrilateral ABCD with $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{CB}$.

To Prove: ABCD is a parallelogram.



1. Draw diagonals $\overline{AC}$ and $\overline{DB}$.
2. By means of the S.S.S. Theorem we can prove $\triangle ABC \cong \triangle CDA$ and $\triangle DAB \cong \triangle BCD$.
3. By corresponding parts $\angle a \cong \angle b$ and $\angle c \cong \angle d$.
4. Then by Theorem 9-4, $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$.
5. By the definition of a parallelogram ABCD is a parallelogram.

Proof of Theorem 9-19:



Given: Quadrilateral ABCD with $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \parallel \overline{CD}$.

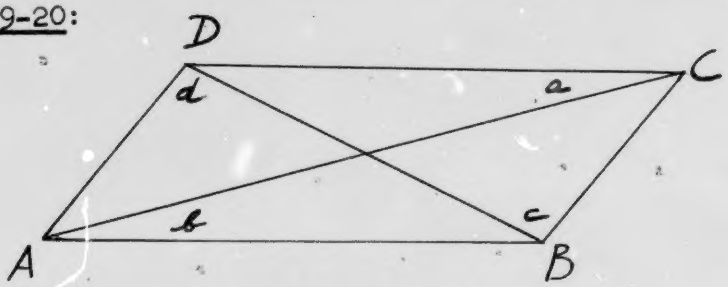
To Prove: ABCD is a parallelogram.

Proof: Since $\overline{AB} \parallel \overline{CD}$, $\angle a \cong \angle b$ by alternate interior angles $\overline{AC} = \overline{CA}$, and $\triangle ABC \cong \triangle CDA$ by S.A.S. Postulate. Then $\angle BCA \cong \angle DAC$ by corresponding parts,

And by Theorem 9-4, $\overline{DA} \parallel \overline{BC}$.
Since both pairs of opposite sides of this quadrilateral are parallel, ABCD is a parallelogram.

Alternate Proof of Theorem 9-19 after getting $\triangle ABC \cong \triangle CDA$ then $\overline{DA} \cong \overline{BC}$ and by Theorem 9-18 ABCD is a parallelogram.

Proof of Theorem 9-20:



Given: Quadrilateral ABCD with diagonals $\overline{DB}$ and $\overline{AC}$ bisecting each other at P.

To Prove: ABCD is a parallelogram.

1. $\overline{DP} \cong \overline{PB}$ $\overline{AP} \cong \overline{PC}$	1. Given.
2. $\angle CPB \cong \angle DPA$ $\angle DPC \cong \angle BPA$	2. Vertical angles are congruent.
3. $\triangle DPC \cong \triangle BPA$ $\triangle CPB \cong \triangle APD$	3. S.A.S. Postulate.
4. $\overline{AB} \cong \overline{CD}$ $\overline{AD} \cong \overline{CB}$	4. Corresponding parts.
5. ABCD is a parallelogram.	5. Theorem 9-18.

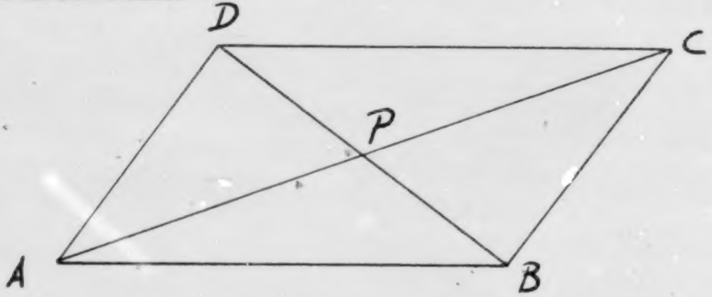
270 Theorem 9-22: By Theorem 9-16 the consecutive angles of a parallelogram are supplementary, and since one angle is a right angle its supplement must be a right angle. Two successive applications of Theorem 9-16 will establish that the other two angles

are right angles. A shorter way would be one application of the theorem that opposite angles are congruent.

Theorem 9-22 gives us an efficient way to prove that a quadrilateral is a rectangle. First prove that it is a parallelogram and then prove that one angle is a right angle.

In some texts a rectangle is defined as follows: "If one angle of a parallelogram is a right angle then the figure is a rectangle". If this definition is used then you would want to prove the Theorem: If one angle of a parallelogram is a right angle then all four angles are right angles. Using the definition of a rectangle and this theorem together we see that a rectangle has four right angles.

Proof of Theorem 9-23:

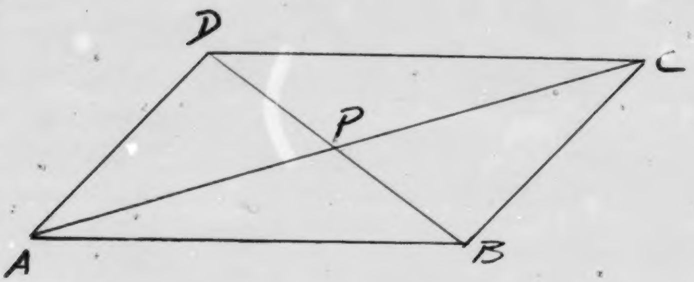


Given: Rhombus ABCD with diagonals AC and BD.

To Prove: $\overline{AC} \perp \overline{BD}$.

1. $\overline{DP} \cong \overline{PB}$	1. Diagonals of a parallelogram bisect each other.
2. $\overline{DC} \cong \overline{BC}$	2. Definition of rhombus.
3. $\overline{PC} = \overline{PC}$	3. Identity.
4. $\triangle DPC \cong \triangle BPC$	4. S.S.S.
5. $\angle CPD \cong \angle CPB$	5. Corresponding parts.
6. $\overline{DP} \perp \overline{AC}$	6. Definition of perpendicular.

Theorem 9-24.

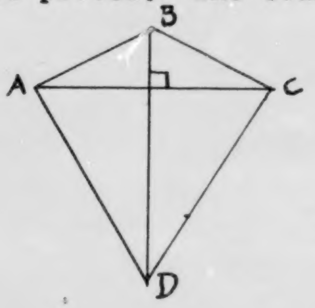


By proving $\Delta BPC \cong \Delta DPC \cong \Delta DPA \cong \Delta BPA$ by S.A.S., we can establish that quadrilateral ABCD is a rhombus, since $\overline{DC} \cong \overline{CB} \cong \overline{BA} \cong \overline{AD}$.

270 After the class has become familiar with the properties of quadrilaterals stated on the previous pages you might propose the following two problems for them to work. Neither of these two problems can be solved since there is a counter-example (an example satisfying all of the given conditions that does not give the desired result) for each one.

(1) Given quadrilateral ABCD such that $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \cong \overline{BC}$, prove this quadrilateral is a parallelogram. Do not inform the students that this cannot be proved. Let them search for themselves for a while and perhaps realize that the counter-example is an isosceles trapezoid. This figure satisfies all of the given data, but certainly is not a parallelogram.

(2) Given a quadrilateral ABCD such that the diagonals are perpendicular to each other. Prove that the quadrilateral is a rhombus (or a square). This statement, also, cannot be proved. The counter-example is a figure that looks like this: It consists of a kite shaped arrangement with two non-congruent isosceles triangles having the same base fitted together as in the figure.



276 Caution the students that the statement of Theorem 9-25 does not say that the segments intercepted on one transversal are congruent respectively to segments intercepted on another transversal. The segments of any one transversal are congruent to each other.

A word about the Problem Sets.

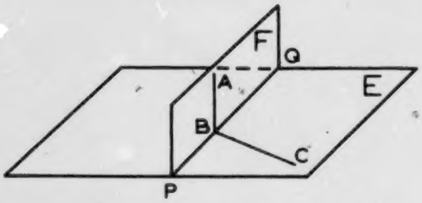
Some of the problems from here on will be labeled "Theorem" with no theorem number following. These theorems will be very useful in some of the problems that follow in the same set and may also be useful in later problem sets. Even if a Theorem in the problem set is not assigned, the student should be permitted to use the Theorem in any subsequent exercise, if he wishes to do so. This is not absolutely necessary, but by using the Theorems the proofs will be much easier and shorter.

Chapter 10

In this chapter you will see a very strong analogy between the material concerning parallel lines in a plane as described in Chapter 9, and the discussion of parallel planes in space. By inserting the word "line" for "plane" in the statements of Theorems 10-2 through 10-5 we get statements that are equivalent to theorems that appeared in Chapter 9. In some cases the proofs are a bit more involved, since we are working in space and not just in one plane.

292 Notice in the definition of dihedral angles, that we cannot just speak of the union of two half-planes, but that we must include their common edge in the union. This is because a half-plane does not contain its edge.

295 Proof of Corollary 10-6-1: If a line is perpendicular to a plane, any plane containing this line is perpendicular to the given plane.



Given: $\overleftrightarrow{AB} \perp E$, F contains $\overleftrightarrow{AB}$

To Prove: $F \perp E$.

In E draw $\overleftrightarrow{BC} \perp \overleftrightarrow{PQ}$. Since $\overleftrightarrow{AB} \perp E$, then by the definition of a line perpendicular to a plane, $\overleftrightarrow{AB} \perp \overleftrightarrow{PQ}$, likewise $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$. Hence $\angle ABC$ is the plane angle of A - PQ - C, since $\overleftrightarrow{AB}$ and $\overleftrightarrow{CB}$ are perpendicular to PQ at B. Therefore we see that $\angle ABC$ is a right angle and then $F \perp E$ by the definition of perpendicular planes.

Proof of Corollary 10-6-2: If two planes are perpendicular, any line in one of them, perpendicular to their line of intersection is perpendicular to the other.

Given: $F \perp E$, $\overleftrightarrow{AB} \perp \overleftrightarrow{PQ}$

To Prove: $\overleftrightarrow{AB} \perp E$.

Using the figure above, in E draw $\overleftrightarrow{BC} \perp \overleftrightarrow{PQ}$. Then by the definition of a plane angle, $\angle ABC$ is the plane angle of $A - PQ - C$. $F \perp E$ by hypothesis, then $\angle ABC$ is a right angle, and $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$. Since it was given that $\overleftrightarrow{AB} \perp \overleftrightarrow{PQ}$, we now have $\overleftrightarrow{AB}$ perpendicular to two lines in E passing through its foot, hence by Theorem 8-3 $\overleftrightarrow{AB} \perp E$.

301 The answer to the "Why?" is that the perpendicular to E through S is parallel to $\overleftrightarrow{PQ}$ (by Theorem 10-4), and therefore must lie in the plane determined by P, Q, and S. This is the plane E'.

Chapter 11.

The concept of area is not a new one to students. From a very early stage they have developed an intuitive notion of what an area is. However, in this treatment, we must introduce a new unit of measure, the square unit, and postulate the formula for finding the area of one region, a rectangular region.

308 Note that in the figures on this page it is intuitively clear that the areas of the regions can be found by dividing them up into smaller triangular regions, and that the area of the total region is independent of the manner in which the triangular regions are formed.

309 Postulate 17 establishes a correspondence between the areas of polygonal regions and the positive numbers. That is, to each polygonal region R there corresponds one and only one positive number, A, called the area of R. This postulate guarantees a correspondence that goes from polygonal regions to positive numbers. That is, given a polygonal region, we know there is a unique positive number that is its area. The question arises, is every positive number the correspondent of some the area of some polygonal region? Intuitively we feel that to each positive number A, there corresponds at least one polygonal region with A as its area, and in fact, we suspect there are many, many such regions. However, in spite of our feeling, Postulate 17 tells us nothing about this case. After Postulate 20 we would be able to prove that our intuitive feeling is correct, but we cannot do it on the basis of Postulates 1 through 17. After we have Postulate 20, we could draw a rectangle with one side of length A and the other side of length 1. Then,

by Postulate 20, the area is A. Hence we have *at least one* polygonal region that corresponds to any positive number. By using Postulate 19, we could cut up the rectangle into triangles and reassemble them and get many, many polygonal regions with area A.

312 In order to get on to the business of finding areas of various regions we postulate the area of a rectangle to be A = bh. It is not an easy task to *prove*, at this level, that areas behave as we would expect them to behave. The usual process of cutting up a rectangle into little squares and finding the formula by counting these squares is *not* a proof, since the formula for the area of each of these little squares has not been proved. Moreover the process won't work in general if the sides are not rational numbers, and a complete treatment would require the theory of limits. The most practical and sound method at this level is therefore to postulate the area of one type of region and develop the formulas for the areas of other regions on the basis of this postulate.

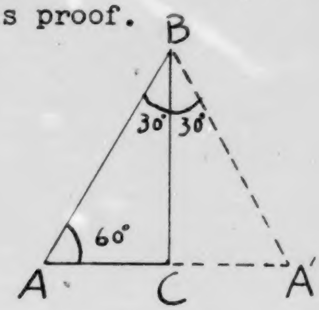
318-322 Notice that, after postulating the area of a rectangle, we proceed to develop our formulas for areas in the following manner: right triangles, which then permit us to work with any triangle (see page 318), parallelograms, and trapezoids. Of course our postulate permits us to find the area of a square, since it is merely an equilateral rectangle. At this point we have the machinery to find the area of any polygonal region, by just chopping it up into a number of triangular regions, and finding the sum of the areas of these triangular regions.

320 Note that in the discussion of the area of a triangle, it does not matter which altitude and base we consider, just so long as we

work with a base and the corresponding altitude.

321 Just as with triangles, we may work with either side and the corresponding altitude of a parallelogram.

329 We have here a very simple proof of the Pythagorean Theorem. The proof depends upon the properties of the areas of triangles and squares. Notice how Postulate 19 is used in this proof.



338 Proof of Theorem 11-10:

Given: Right $\triangle ABC$ with $m\angle B = 30$ and $m\angle A = 60$

To Prove: $AB = 2(AC)$

Proof: Draw in the construction lines as shown in the figure. Then $\triangle AA'B$ is equilateral. Hence $AC = \frac{1}{2}AA'$ and $AA' = AB$; therefore $AC = \frac{1}{2}AB$ and $AB = 2(AC)$. You may wish to establish the following additional property, namely $BC = \sqrt{3}(AC)$.

Let $AC = b$, then by the above property, $AB = 2b$. By the Pythagorean Theorem: $(BC)^2 = 4b^2 - b^2$

$$(BC)^2 = 3b^2$$

$$BC = b\sqrt{3} \quad \text{or} \quad BC = \sqrt{3}\,(AC).$$

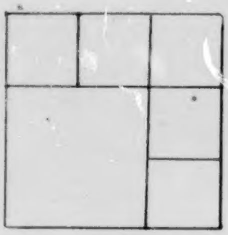
In words: In a 30 - 60 right triangle the side opposite the 30° angle is one-half of the hypotenuse and the side opposite the 60° angle is $\sqrt{3}$ times the side opposite the 30° angle. Given any side of a 30 - 60 right triangle, we can find the remaining sides by using the above properties.

338 Proof of Theorem 11-11:

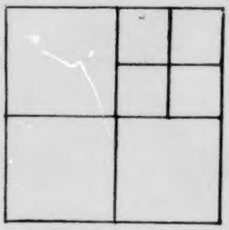
Let each leg of the right triangle have measure a. Then by the Pythagorean Theorem, $c^2 = 2a^2$, or $c = a\sqrt{2}$, which gives us $\frac{c}{a} = \sqrt{2}$. These two theorems have many numerical applications and students should be quite familiar with them before attempting the problems that follow.

Note to the teacher: At this point in the text you may wish to proceed directly to Chapter 16, Plane Coordinate Geometry, and later return to the remaining chapters.

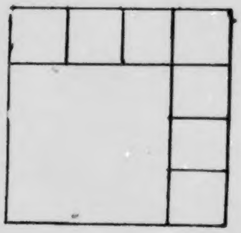
Here is a problem that might be interesting to the class. It has to do with cutting up a square into a certain number of smaller squares, not necessarily equal in area. We will talk of a number k, as being "acceptable" if a square can be subdivided into k squares. For example, given any square we can divide it into 4 squares, but not into 2, 3, or 5. Try it. Below are some diagrams showing how a square may be divided into 6, 7, and 8 smaller squares:



$k = 6$



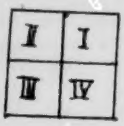
$k = 7$

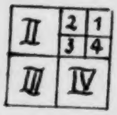


$k = 8$

We may ask is there some pattern or some number k, above which this will always be possible. Actually any $k \geq 6$ will always be acceptable. Let us now show that if a square can be divided into k smaller squares, then it can be divided into $k + 3$ smaller squares:

Imagine that we have already divided a square into k square regions. Now, split one of the squares into 4 smaller squares by bisecting the sides. In this process we have lost one larger square and gained four smaller ones, thus gaining three. Let us illustrate using $k = 4$.





After dividing the original square into 4 smaller squares, we take one of them, and divide it into 4 equal squares. Instead of having 4 squares from the first division we have only three, and now have 4 additional ones giving a total of 7. Since we know that $k = 6$, 7, and 8 are acceptable, and that we can get $k + 3$ squares from any division, we can form the following sequences:

$$6, 9, 12, 15, \ldots,$$
$$7, 10, 13, 16, \ldots,$$
$$8, 11, 14, 17, \ldots,$$

Hence all $k \geq 6$ are acceptable.

Chapter 12.

In Chapter 6 we explored the concept of congruence, which encompassed the idea of a one-to-one correspondence between angles and sides of two triangles such that corresponding sides and corresponding angles were congruent. In this chapter we will talk of a correspondence between triangles such that corresponding angles are congruent and the ratios of corresponding sides are equal to a constant. This correspondence is called a similarity. After a discussion of proportions, there appears a proof of the fundamental similarity theorem for triangles that is different from the usual one given. This proof is not new; quite the contrary. It appeared in a text-book known to be over 100 years old. More will be said about it later.

349 Note carefully the notation for proportional sequences, namely "$\ldots \sim \ldots$". You will see that each number on the right of this sign is found by multiplying the corresponding number to the left of the sign by some constant, namely, the constant of proportionality. When we talk later of proportional segments, we will be concerned with the lengths of one set of segments and the corresponding lengths of another set of segments. The lengths of the segments in one set will be a constant times the corresponding lengths in the other set. The material on this page should be noted carefully, for instance the manner in which $a, b, c \sim a', b', c'$ is used to set up fractions to determine the constant of proportionality.

$a, b, c \sim a', b', c'$ means $\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = k$, which also implies:

$a' = ka, \quad b' = kb, \quad c' = kc.$

353 To say that corresponding sides of similar triangles are pro-

portional means that the length of each side of one triangle can be found by multiplying the length of each corresponding side of the other triangle by the same number.

357 The student is expected to call upon his algebra in working with proportionalities. If three parts of a proportion are given and it is required to find a fourth, we need no statements about the properties of proportions in addition to those we already have. The quantities used in proportions of this type are numbers, and the algebra of fractional equations will enable him to do all that is required.

358 The geometric mean of two positive numbers, a and c, is the positive number b, such that: (1) $a, b \sim b, c$.
Algebraically this means (2) $\frac{b}{a} = \frac{c}{b}$.
Since a, b, and c are positive, this is equivalent to
(3) $\frac{a}{b} = \frac{b}{c}$ which can be written (4) $b, c \sim a, b$.
You may recognize that in (3), b is what has been called in some textbooks, the mean proportional between a and c. We speak of this as the geometric mean of a and c, and $b = \sqrt{ac}$. Then "geometric mean" and "mean proportional" are names for the same thing, and we prefer to use "geometric mean" in this text. In mathematics there are such things as harmonic and arithmetic means that do not arise from proportions, and we have used "geometric mean" because it arose historically in a geometric construction.

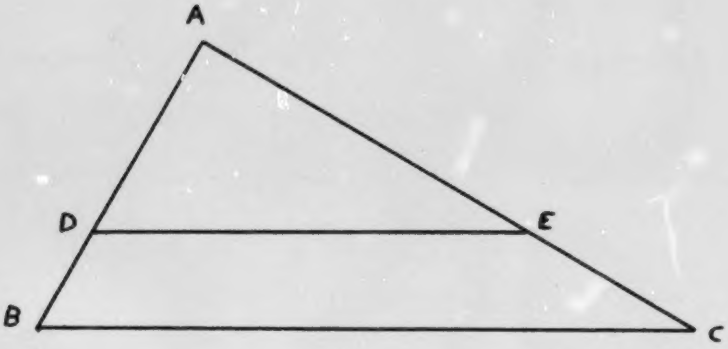
362 We come now to the proof of the basic similarity theorem. The chapter on areas comes before the chapter on similarity because we shall use this concept of area in the proof of the basic similarity theorem. Conventional proofs of this theorem have had to contend

with (1) a relatively unconvincing division of the sides of a triangle by a series of parallel lines, and (2) the problem of what to do about the situation when the ratio of the length of a segment to the length of a side containing that segment is not a rational number (commensurable vs. the incommensurable case). It has usually been the practice to prove this theorem for the commensurable case and mention the other possibility. Here is a proof that does not require the use of these two cases. However, it appears too difficult to give to high school students, and appears here for examination by the teacher and for use with those students who may be interested.

The proof in the text also avoids the difficulty of incommensurables since it is based on areas and uses Postulate 20. Postulate 20 makes it possible for us to avoid a separate discussion of the incommensurable case. It is believed that this proof is within the grasp of the high school student. It was found in a textbook, published in 1855, and is the work of the noted French mathematician, A. M. Legendre.

Here is the Proof of the Basic Similarity Theorem, not using area, which includes the incommensurable case without specifically saying so.

Theorem: If a line intersects two sides of a triangle in distinct points and is parallel to the third side then it cuts off segments which are proportional to these two sides.

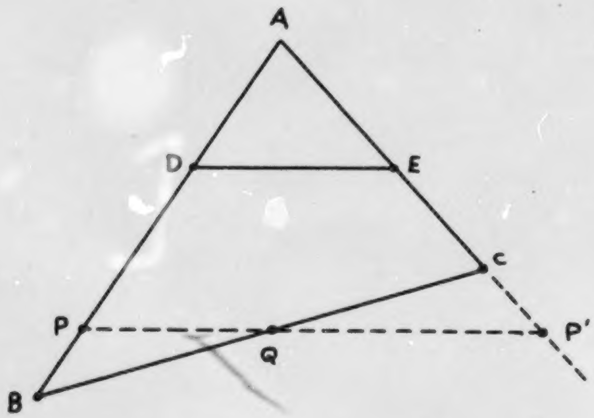


Restatement. Let $\triangle ABC$ be a triangle. Let D and E be points of $\overline{AB}$ and $\overline{AC}$. If the lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{DE}$ are parallel, then

$$\frac{AB}{AD} = \frac{AC}{AE} .$$

The proof is going to be indirect. We are going to show that if the proportionality does not hold, then $\overleftrightarrow{BC}$ and $\overleftrightarrow{DE}$ are not parallel. If we can show this, then the theorem will be proved.

The main idea of the proof is this. If the proportionality does not work, then one of the segments $\overline{AB}$ and $\overline{AC}$ is too long. Suppose that it is $\overline{AB}$ that is too long, like this:



We will prove that there is a point P, between D and B, and a point P', on the opposite side of $\overleftrightarrow{BC}$ from A, such that

$\overleftrightarrow{PP'}$ and $\overleftrightarrow{DE}$ are parallel. Since P and P' are on opposite sides of $\overleftrightarrow{BC}$, the segment $\overline{PP'}$ intersects $\overleftrightarrow{BC}$ in a point Q, as in the figure. This is impossible, because it means that there are two parallels to $\overleftrightarrow{DE}$ through Q, and this contradicts the Parallel Postulate.

Let us now get on with the proof, and see how such points P and P' are to be found.

Proof. Suppose that the fractions $\frac{AB}{AD}$ and $\frac{AC}{AE}$ are not equal, and suppose that the first fraction is the one that is bigger. Thus

$$\frac{AB}{AD} > \frac{AC}{AE} .$$

The first step in the proof is to find a point F on $\overline{AB}$ that makes the proportionality work. That is, we want a point F on $\overline{AB}$ so that

$$\frac{AF}{AD} = \frac{AC}{AE} .$$

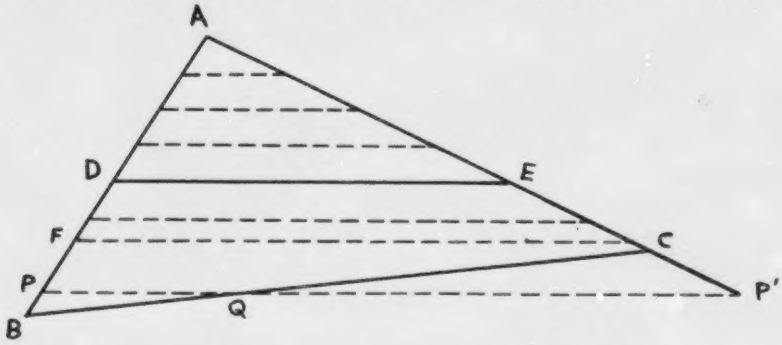
This equation means that

$$AF = AD \cdot \frac{AC}{AE} ,$$

and by Theorem 4-3 there is such a point, on the ray $\overrightarrow{AB}$. Since $\frac{AC}{AE} > 1$, then AF > AD and since

$$AB > AD \cdot \frac{AC}{AE} , \quad \text{then } AB > AF, \text{ and}$$

F must lie between D and B, as in the figure below.



Let us divide $\overline{AD}$ into congruent segments, laid end to end, so that each of these segments is shorter than $\overline{BF}$. (In the figure, $\overline{AD}$ has been divided into four segments.) Now if we lay off more segments of the same length, we will reach a point P which is between F and B. Through the end points of these short segments we take lines parallel to $\overleftrightarrow{DE}$. (These are the dotted lines in the figure.) Let P' be the point where the line through P, parallel to $\overleftrightarrow{DE}$, intersects the line $\overleftrightarrow{AC}$.

These are the points P and P' that we are looking for. We already know that P is on the same side of $\overleftrightarrow{BC}$ as A, because P is between B and D. And we know that $\overleftrightarrow{PP'}$ and $\overleftrightarrow{DE}$ are parallel. Therefore, to complete the proof, all that we need to show is that P' and A are on opposite sides of $\overleftrightarrow{BC}$. We will do this by showing that $AP' > AC$.

We know that the dotted lines cut off congruent segments on $\overleftrightarrow{AB}$. By Corollary 9-25-1, the dotted lines also cut off congruent segments on $\overleftrightarrow{AC}$. Therefore

$$\frac{AP}{AD} = \frac{AP'}{AE},$$

because each of these fractions is equal to the number of little segments in $\overline{AP}$ divided by the number of little segments in $\overline{AD}$ We also know that

$$\frac{AF}{AD} = \frac{AC}{AE}.$$

By division, we get

$$\frac{AP}{AF} = \frac{AP'}{AC}.$$

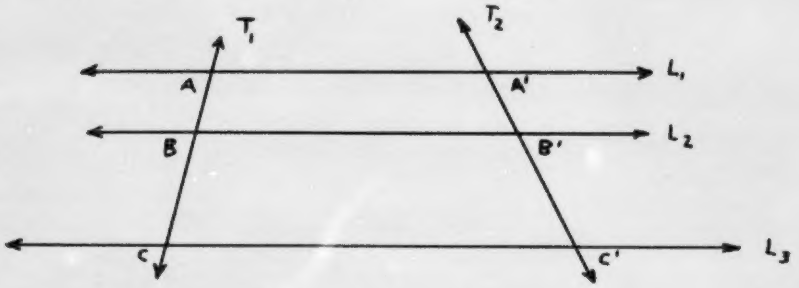
The first fraction is >1, because $AP > AF$. Therefore the second fraction is >1. Therefore $AP' > AC$. Therefore A and P' are on opposite sides of $\overleftrightarrow{BC}$.

Since P and P' are on opposite sides of $\overleftrightarrow{BC}$, it follows that $\overleftrightarrow{PP'}$ intersects $\overleftrightarrow{BC}$ in a point Q. But this is impossible, because it means that through the point Q there are two lines $\overleftrightarrow{PP'}$ and $\overleftrightarrow{BC}$, each of them being parallel to $\overleftrightarrow{DE}$. This contradicts the Parallel Postulate. We arrived at this contradiction by assuming that the theorem was false. Therefore the theorem is true, which was to be proved.

In connection with this proof of the Basic Similarity Theorem we have again assumed that $\overleftrightarrow{DE}$ intersects $\overline{AC}$ at a point between A and C. The figure tells us that this appears to be so, and it is, but we have not yet proved this to be so. To do this we need to prove a theorem:

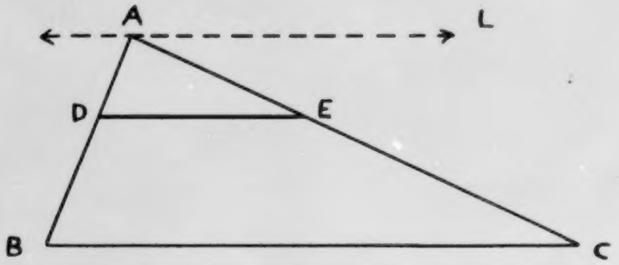
"Given transversals T_1 and T_2 intersecting three parallel lines L_1, L_2, L_3 in points A, B, and C and A', B', and C' respectively as shown in the figure. If B is between A and C then B' is between A' and C'.

Stated in a more formal manner this theorem becomes: "Betweeness is preserved under parallel projection".



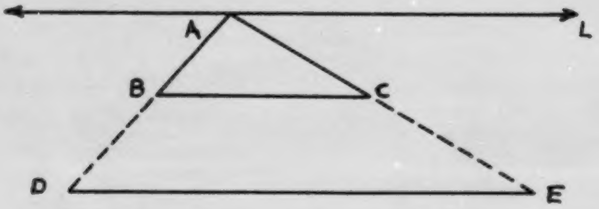
Proof: Since $L_1 \parallel L_2$, then the segment $\overline{AA'}$ cannot intersect L_2 and hence A and A' are on the same side of L_2. Likewise, since $L_3 \parallel L_2$, then the segment $\overline{CC'}$ cannot intersect L_2 and C and C' are on the same side of L_2. Since B is between A and C by hypothesis, segment $\overline{AC}$ intersects L_2 at B; hence, A and C are on opposite sides of L_2. Since A' and A are in the same half plane determined by L_2 and C' and C are in the same half plane and A and C are in opposite half planes then it follows that A' and C' are in opposite half planes determined by L_2. Hence $\overleftrightarrow{A'C'}$ meets L_2 in a point which must be B', since B' is the intersection of $\overleftrightarrow{A'C'}$ and L_2. Therefore B' is between A' and C'.

Now we may consider this theorem as it applies to the Basic Similarity Theorem. Let us consider △ABC and point D on $\overline{AB}$ between A and B. What we want to show is that a line through D parallel to $\overleftrightarrow{BC}$ intersects $\overline{AC}$ at some point E which is between A and C.



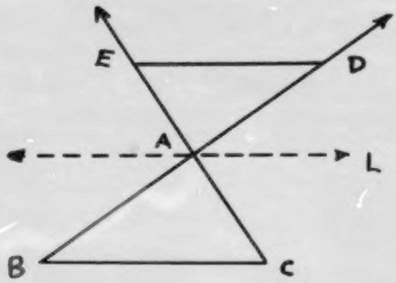
If we draw a line L through A parallel to $\overleftrightarrow{BC}$, we immediately have the same situation as in the theorem above, namely the case of two transversals ($\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$) intersecting three parallel lines. Likewise, if we consider the problem when B is between A and D, we would want to know if a line through D parallel to $\overleftrightarrow{BC}$ intersects $\overleftrightarrow{AC}$ at a point E such that C is between A and E.

Consider the figure below:



Again, if we draw a line L parallel to $\overleftrightarrow{BC}$ through A, then we have the same situation as in the theorem.

There is a third case that we might consider. Namely, if E is a point on $\overleftrightarrow{AC}$ such that A is between E and C, then a line through E parallel to $\overleftrightarrow{BC}$ intersects $\overleftrightarrow{AB}$ in a point D such that A is between D and B. The figure below illustrates this case:



By drawing a line L through A parallel to $\overleftrightarrow{BC}$ we again have a special case of the original theorem.

This Parallel Projection Theorem is tacitly assumed in Theorem 12-2 when we assumed that E was between A and C'.

371 In the proofs of Theorems 12-3, 12-4, and 12-5 we have drawn the figure as if AB > DE and used this fact except in Theorem 12-3 the case when AB = DE was discussed. In case AB < DE a similar proof would be given but this time E' would be on $\overrightarrow{DE}$ and DE' = AB.

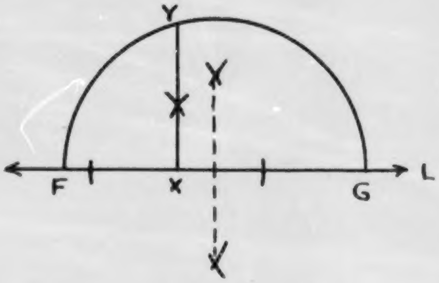
374 In the proof of Theorem 12-5, it might be advisable to point out to the students the general plan of the proof. First prove $\triangle ABC \sim \triangle AE'F'$ by A.S.A. Similarity Theorem, then prove

$\triangle AE'F' \cong \triangle DEF$ by the S.S.S. Theorem, then prove $\triangle ABC \sim \triangle DEF$ by the A.A.A. Similarity Theorem.

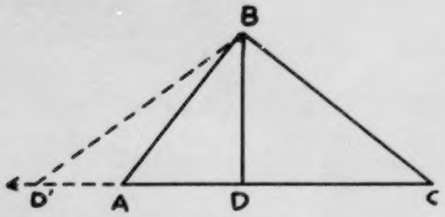
388 One consequence of Theorem 12-6 is that it gives us a means whereby we can construct a segment whose length is the geometric mean of the lengths of two given segments. One additional step in the proof is required. If $\triangle ACD \sim \triangle CBD$, then it follows that $\frac{AD}{CD} = \frac{CD}{BD}$. From this we see that CD is the geometric mean between AD and BD. Now to find a way to construct this geometric mean. From the proof of this theorem, it appears that we must have a right triangle with the altitude drawn to the hypotenuse such that the segments into which it divides the hypotenuse are congruent to the two given segments. We need to know but one more thing. Namely, that an angle inscribed in a semi-circle is a right angle; this is Corollary 13-6-1 in the next chapter.

"Given two segments of length a and b as follows, then construct a segment whose length c is the geometric mean of a and b."

Take any line L, and on L lay off two segments FX and XG end to end, of lengths a and b respectively. Describe a semi-circle about $\overline{FG}$ as a diameter. At X, the common end-point of the two segments, erect a perpendicular intersecting the semi-circle at some point Y.



XY, then, is the required geometric mean. By drawing $\overline{FY}$ and $\overline{GY}$, we can immediately prove by Theorem 12-6, that XY is the geometric mean between a and b.



In any right triangle, the altitude to the hypotenuse intersects the hypotenuse in a point between the end points of the hypotenuse.

There are 5 possible cases:

(1) D = A

(2) D = C

(3) A is between D' and C.

(4) C is between D' and A.

(5) D is between A and C.

We would like to show that cases (1), (2), (3), and (4) are impossible which leaves case (5) as the required result.

Case (1) is impossible because triangle BDC then would have two right angles, one at B and one at D.

Case (2) is impossible for a similar reason as used in Case (1).

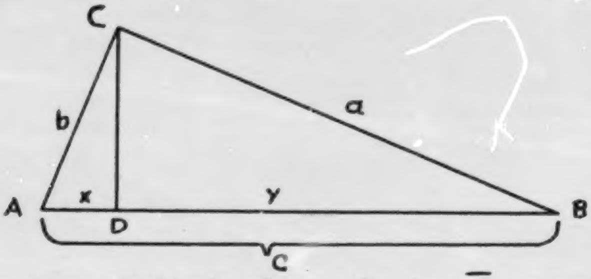
We proceed to prove that Case (3) is impossible.

A similar proof will show that Case (4) is impossible, hence Case (5) holds as was to be proved.

Proof of Case (3): Let us assume that the altitude from B to $\overleftrightarrow{AC}$ intersects $\overleftrightarrow{AC}$ at a point D', such that A is between D' and C. Then, since $\overline{BD'}$ is an altitude, $\angle BD'C$ is a right angle. $\angle BAD$ is one of the acute angles of $\triangle ABC$. $\angle BAD$ is also an exterior angle of $\triangle BD'A$ and therefore should be greater than $\angle D'$. But this cannot be so, since $\angle D'$ is a right angle, hence the altitude from B must intersect the hypotenuse at some point D, such that D is between A and C.

389 Once we have proved Theorem 12-6, it is now possible to prove the Pythagorean Theorem in the conventional manner using similar triangles. This has not been done in the text, however, since the theorem has been proved once by areas. If time permits, it might be illuminating to the class to let them see the following proof, and realize that there is more than one way to attack a mathematical problem.

Theorem: Given a right triangle, with legs of length a and b and hypotenuse of length c. Then $a^2 + b^2 = c^2$.



Proof: Let $\overline{CD}$ be the altitude from C to $\overline{AB}$, as in Theorem 12-6. Let $x = AD$ and let $y = DB$, as in the figure. The scheme of the proof is simple. (1) First we calculate x in terms of b and c, using similar triangles. (2) Then we calculate y in terms of a and c, using similar triangles. (3) Then we add x and y, and simplify the resulting equation, using the fact that $c = x + y$.

(1) Since $\triangle ACD \sim \triangle ABC$,

we have $\frac{x}{b} = \frac{b}{c}$.

Therefore $x = \frac{b^2}{c}$.

(2) Since $\triangle CBD \sim \triangle ABC$, we have $\frac{y}{a} = \frac{a}{c}$.

Therefore $y = \frac{a^2}{c}$.

(3) Thus we have $x + y = \frac{a^2 + b^2}{c}$.

But $c = x + y$.

Therefore $c = \frac{a^2 + b^2}{c}$,

and $a^2 + b^2 = c^2$, which was to be proved.

Chapter 13

In this Chapter we will discuss the fundamental properties of circles, spheres, tangent lines and planes, secant lines and planes, angle and arc measurement in a plane. You will note that following the fundamental theorems on circles, there is a corresponding section concerning spheres, and probably nowhere else is the analogy between plane and space geometry so strong as it is here. Essentially the same proofs work for the sphere as the circle, as relates to tangent and secant lines and planes.

406 We will frequently use the definitions of interior and exterior of a circle in our proofs. Suppose a point is in the plane of the circle. To show it lies in the *interior* of a circle merely requires that we establish that its distance from the center of the circle is *less than* the radius, and to be in the *exterior* of a circle its distance from the center of the circle must be *greater than* the radius.

407 The same discussion applies to the interior and exterior of a sphere as to a circle.

The argument shows there are at most two points X
413 which can lie on both L and the circle, and it locates
these points on L by finding their distance from Q. To show explicity that there *are* two points common to both, we merely have to show that the two points X are on the circle. Thus we find the distance PX from each point to the center:

$$PX = \sqrt{PQ^2 + QX^2} = \sqrt{PQ^2 + r^2 - PQ^2} = \sqrt{r^2} = r .$$

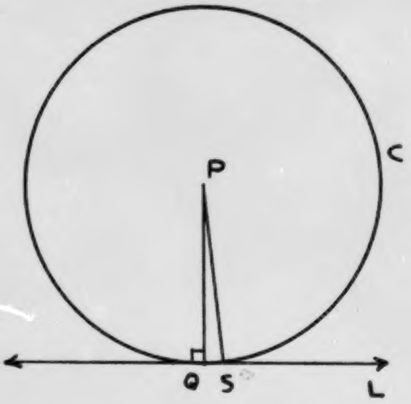
414 Proof that if $PQ > r$, L does not intersect the circle C. The shortest distance from a point P to the points of line L is the

length of the perpendicular segment from P to L. If Q is on L, $PQ \perp L$ and $PQ > r$, then for Q', any point of L, $PQ' > r$, hence every point of L is in the exterior of the circle and the line does not intersect the circle.

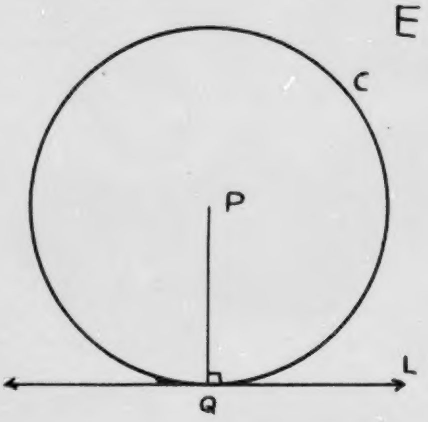
Before proving the Corollaries, let us see exactly what was established in Theorem 13-1. Since Q is the foot of the perpendicular from the center of the circle C to the line L, if case (1) holds (L intersects C in the point Q alone) we see that L is a tangent to the circle and L is perpendicular to the radius $\overline{PQ}$. We see also that if L intersects the circle in a point different from Q, that L intersects the circle in exactly two points which are equidistant from Q.

Proof of Corollary 13-1-1: Any line tangent to C is perpendicular to the radius drawn to the point of contact.

Proof: Let L be a tangent to C at point S. Draw the radius $\overline{PS}$. Let Q be the foot of the perpendicular from P to L. If $Q \neq S$, then L intersects C in exactly 2 points and this contradicts the hypothesis that L is tangent to C at S. Therefore the point Q must be the point S, hence the tangent L is perpendicular to the radius drawn to the point of contact.

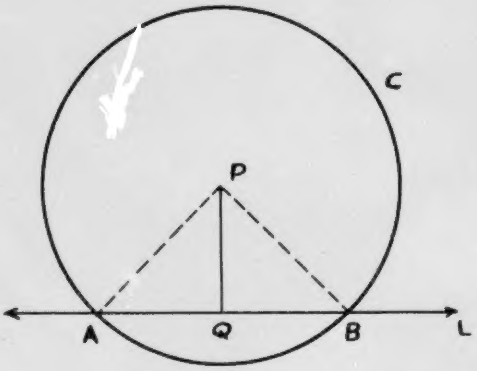


414 Proof of Corollary 13-1-2: Any line in E perpendicular to a radius at its outer end, is tangent to the circle.



Proof: Given a line in E, perpendicular to a radius at its outer end, which is a point on circle C. This point is Q, the foot of the perpendicular from center P to L. Then, by Theorem 13-1, the line intersects the circle in Q alone and is therefore tangent to the circle.

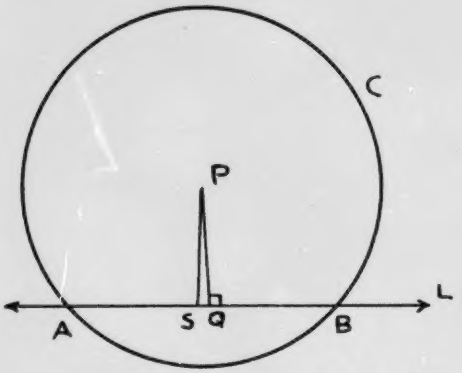
Proof of Corollary 13-1-3: Any perpendicular from the center of C to a chord bisects the chord.



Proof: Consider a chord $\overline{AB}$ of circle C and the line L containing $\overline{AB}$. The line L intersects C in two points A and B. Let Q be the foot of the perpendicular from P to L. The intersection cannot be Q alone. Hence, by Theorem 13-1, A and B are equidistant from Q. Therefore the perpendicular from P to the chord bisects the chord.

Proof of Corollary 13-1-4: The segment joining the center of a circle to the mid-point of a chord is perpendicular to the chord.

Proof: Given chord $\overline{AB}$ of circle C and segment $\overline{PS}$ where P is the center of circle C and S is the mid-point of chord $\overline{AB}$. Let $\overline{PQ} \perp \overline{AB}$ with foot Q. By Corollary 13-1-3, Q is the mid-point of $\overline{AB}$. Since the mid-point of $\overline{AB}$ is unique $(Q = S)$, $\overline{PS}$ is perpendicular to the chord $\overline{AB}$.



414 Proof of Corollary 13-1-5: In the plane of a circle, the perpendicular bisector of a chord passes through the center of the circle.

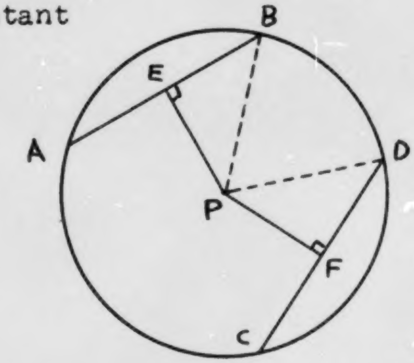
Proof: By Corollary 13-1-4 the segment joining the center of a circle to the midpoint of a chord is perpendicular to the chord, hence the line containing the center of a circle and the mid-point of the chord is a perpendicular bisector of the chord. Since there is only one perpendicular to the chord at its mid-point, the perpendicular bisector of a chord must pass through the center of the circle.

415 Proof of Theorem 13-2: In the same circle or congruent circles, chords equidistant from the center are congruent.

Given: Chords $\overline{AB}$ and $\overline{CD}$, both equidistant from P.

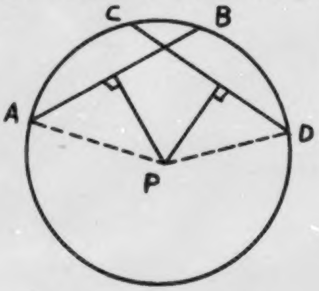
To Prove: $\overline{AB} \cong \overline{CD}$

Proof: Let $\overline{PE} \perp \overline{AB}$ and $\overline{PF} \perp \overline{CD}$ as in the figure. Draw radii $\overline{PB}$ and $\overline{PD}$. Then in right triangles PEB and PFD we have:



(1) $PE = PF$	(1) Given
(2) $\overline{PB} \cong \overline{PD}$	(2) Radii of same or congruent circles are congruent
(3) $\triangle PEB \cong \triangle PFD$	(3) Hypotenuse and Leg Theorem
(4) $EB = FD$	(4) Corresponding parts
(5) $EB = \frac{1}{2}AB$ $FD = \frac{1}{2}CD$	(5) Corollary 13-1-3
(6) $\frac{1}{2}AB = \frac{1}{2}CD$	(6) Substitution
(7) $AB = CD$ or $\overline{AB} \cong \overline{CD}$	(7) Algebra

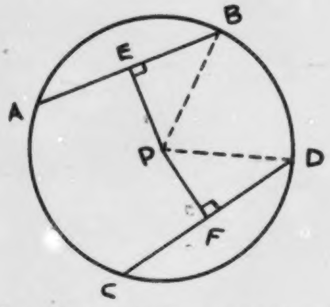
Note that this proof still holds if $\overline{AB}$ intersects $\overline{CD}$ as shown below:



415 Proof of Theorem 13-3: In the same circle or congruent circles, any two congruent chords are equidistant from the center.

Given: Chords $\overline{AB} \cong \overline{CD}$. P is the center of the circle.

To Prove: $PE = PF$ where $\overline{PE} \perp \overline{AB}$ and $\overline{PF} \perp \overline{CD}$ as in the figure.



Proof: Draw radii $\overline{PB}$ and $\overline{PD}$.

(1) $\overline{PB} \cong \overline{PD}$	(1) Radii of same or congruent circles are congruent
(2) $AB = CD$	(2) Given
(3) $\frac{1}{2}AB = \frac{1}{2}CD$	(3) Algebra

(4) $EB = \frac{1}{2}AB$ $FD = \frac{1}{2}CD$	(4) Corollary 13-1-3
(5) $EB = FD$	(5) Substitution
(6) $\triangle PEB \cong \triangle PFD$	(6) Hypotenuse and Leg Theorem
(7) $\overline{PE} \cong \overline{PF}$ or $PE = PF$	(7) Corresponding parts

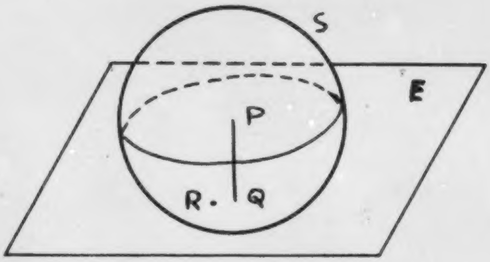
421 Notice how closely the basic theorem on secant and tangent planes, Theorem 13-4, follows the pattern of Theorem 13-1, the basic theorem on secant and tangent lines of a circle.

423 As in the case of Theorem 13-1, the point Q plays a major role in Theorem 13-4 and its corollaries. If plane E intersects S in Q alone then E is tangent to S at Q, and also the segment from P to the point of contact Q is perpendicular to the plane E.

Proof of Corollary 13-4-1: every plane tangent to S is perpendicular to the radius drawn to the point of contact.

Proof: Given: Plane E tangent to S at point R.

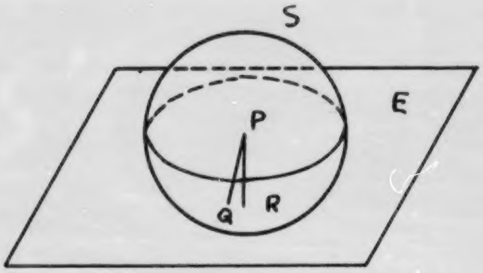
To Prove: Plane E perpendicular to the radius drawn to the point of contact.



Let Q be the foot of the perpendicular from P to E. If Q is not R, then by Theorem 13-4, plane E intersects S in more than one point, which contradicts the hypothesis that E is tangent to S. Hence Q must be R and therefore E is perpendicular to the radius drawn to the point of contact.

423 Proof of Corollary 13-4-2: Any plane perpendicular to a radius at its outer end is tangent to S.

Proof:



Given: Plane E is perpendicular to radius $\overline{PR}$ at R.

To Prove: Plane E is tangent to S.

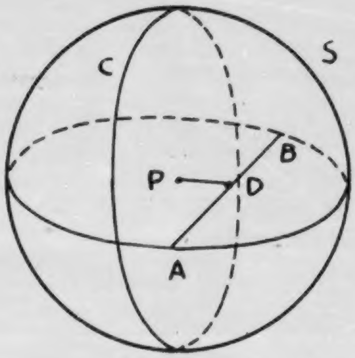
Then R is the foot of the perpendicular to plane E from P. By Theorem 13-4 plane E intersects S only at R, hence E is tangent to S.

424 Proof of Corollary 13-4-4: The segment joining the center to the mid-point of a chord is perpendicular to the chord.

Proof:

Given: Sphere S with D the mid-point of chord $\overline{AB}$. P is the center of S

To Prove: $\overline{PD} \perp \overline{AB}$



If we consider the circle C, the intersection of S and the plane passing through A, P and B, we have exactly the same situation as in Corollary 13-1-4. The proof is identical.

427 Formal definition of semi-circle: Let A and B be the end-points of a diameter of a circle lying in plane E. Then the set of points consisting of A and B and all points of the circle in a given half-plane of E with edge $\overleftrightarrow{AB}$ is a semi-circle. A and B are its end-points.

427 Caution the students that they will be finding the degree measure of arcs and not the length of arcs.

If $\overset{\frown}{AC}$ is a minor arc then the theorem follows from the angle addition postulate. (Postulate 13)

427 It may be noted that if $\overset{\frown}{AC}$ is a semi-circle, the theorem follows immediately from the supplement postulate (Postulate 14). The proof of the general case, though more troublesome, is made to depend upon these two cases. For a complete proof of Theorem 13-5 see Chapter 8 of Studies in Mathematics, Vol. II, School Mathematics Study Group.

428 In the definition of an angle inscribed in an arc, it is important to get across to the student that we are talking about angles inscribed in arcs of circles. Two points separate the circle into two arcs. The student should see that if an angle is inscribed in one of the arcs, the vertex is on that arc and the angle intercepts the other arc. In many geometry texts this is abbreviated to "an angle inscribed in a circle", but this can only mean "inscribed in an arc of a circle", since this is the way it has been defined in the text.

430 Condition (2) for an intercepted arc says, "each side of the angle contains an end-point of the arc". Notice that in the 4th example, in the preceding figures if one side is tangent to the circle, the other side of the angle contains both end-points of the intercepted arc and the tangent contains one end-point. For a discussion of Theorem 13-6 see SMSG Studies in Mathematics, Vol. II.

432 The "Why?" in the first case is the Angle Addition Postulate; in the second case it is Theorem 13-5.

437 Proof of the converse of Theorem 13-7. In the same circle or in congruent circles, if two arcs are congruent, then so also are the corresponding chords.

Proof: Using the figure in the text for Theorem 13-7 we see that:

Given: $\overset{\frown}{AB} \cong \overset{\frown}{A'B'}$

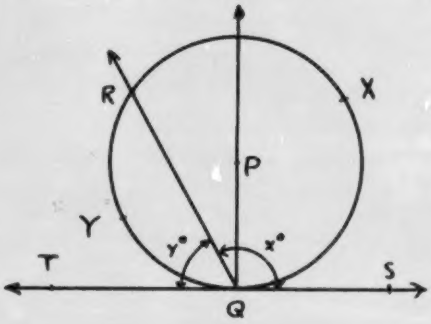
To Prove: $AB = A'B'$

Since $\overset{\frown}{AB} = \overset{\frown}{A'B'}$, $\angle P \cong \angle P'$, and by S.A.S. Postulate we have $\triangle APB \cong \triangle A'B'P'$. Therefore $AB = A'B'$, by corresponding parts.

If $\overset{\frown}{AB}$ and $\overset{\frown}{A'B'}$ are major arcs the same conclusion holds.

If the arcs are semi-circles then the chords are diameters and are congruent.

438 Here is a proof for Theorem 13-9 in the case in which $\angle SQR$ is obtuse.



Given: $\angle SQR$ is obtuse

To Prove: $m\angle SQR = \frac{1}{2} m\, \overset{\frown}{QXR}$

Let $\overrightarrow{QT}$ be the ray opposite to $\overrightarrow{QS}$.

Let x and y be the measures of $\angle SQR$ and $\angle TQR$, as in the figure.

$y = \frac{1}{2} m\, \overset{\frown}{QYR}$ by the case in the text

$x + y = 180$ Postulate 14.

$x = 180 - y$

$x = 180 - \frac{1}{2} m\, \overset{\frown}{QYR}$ substitution

$x = \frac{1}{2}(360 - m\, \overset{\frown}{QYR})$ algebra

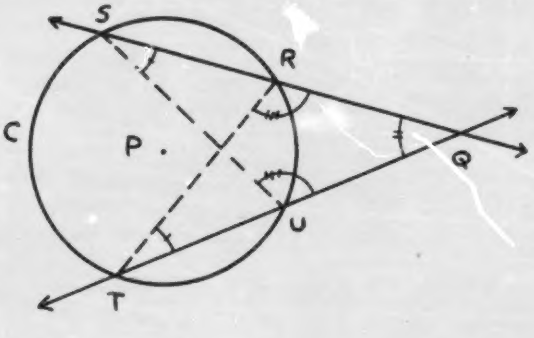
$x = \frac{1}{2} m\, \overset{\frown}{QXR}$ by equation 3 of page 13-16.

444 -447 In the theorems on these pages we will be establishing relationships about the products of the lengths of segments by first establishing a proportion involving these segments using similar triangles.

445 Theorem 13-11 is sometimes stated, "Given a tangent and a secant to a circle from an external point, the length of the tangent is the geometric mean of the length of the secant and the length of its external segment." The reasons in the proof are as follows: (1) Theorem 13-6; (2) Theorem 13-9; (3) Substitution; (4) Corollary 12-3-1 ($\angle Q = \angle Q$ Identity); (5) Corresponding sides of similar triangles are proportional; (6) Multiply both sides by $QR \cdot QT$.

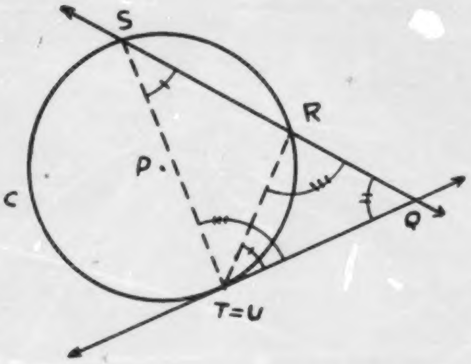
446 Theorem 13-12 stated in words is "If two chords intersect with-
447 in a circle, the product of the lengths of the segments of one equals the product of the lengths of the segments of the other."

If the labeling of the figures for Theorems 13-11, 13-12, and 13-13 is kept consistent as illustrated below $\triangle SQU \sim \triangle TQR$ in each case



$\triangle SQU \sim \triangle TQR$

Theorem 13-11 $QR \cdot QS = QU \cdot QT$



$\triangle SQU \sim \triangle TQR$

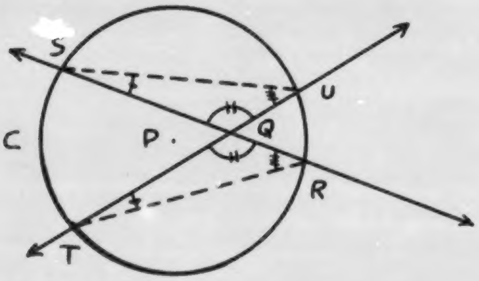
Notice that we used U as the name of the point of contact of the tangent $\overleftrightarrow{QT}$ in $\triangle SQU$ and used T as the name in $\triangle TQR$. Theorem 13-12 $QR \cdot QS = QT^2$

$QR \cdot QS = QU \cdot QT$

but since $T = U$ $QR \cdot QS = QU^2 = QT^2$

$\triangle SQU \sim \triangle TQR$

Theorem 13-13 $QR \cdot QS = QU \cdot QT$



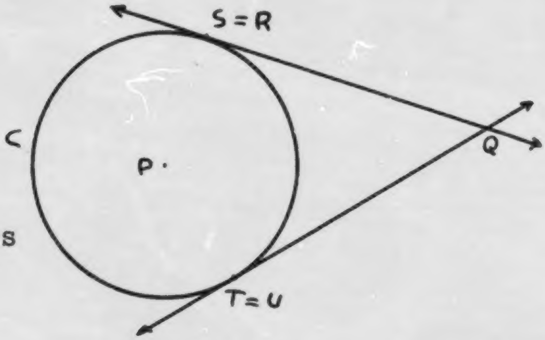
$QR \cdot QS = QU \cdot QT$

Since R = S and T = U we get

$QR \cdot QR = QT \cdot QT$

$QR^2 = QT^2$

Since QR and QT are positive numbers we have Theorem 13-10 QR = QT

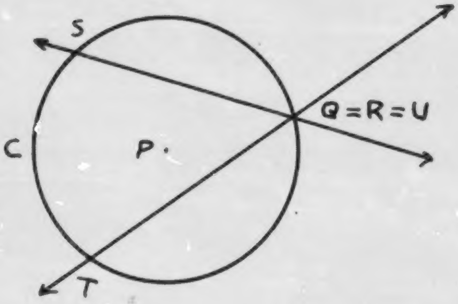


$QR \cdot QS = QU \cdot QT$

Since Q = R = U then QR = 0 and QU = 0, hence $0 \cdot QS = Q \cdot QT$

$$0 = 0$$

and this is a trivial result but the pattern $QR \cdot QS = QU \cdot QT$ still holds.

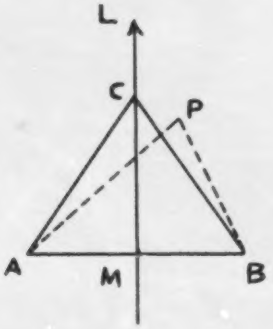


CHAPTER 14

455 In the discussion of the circle as a locus, it might be helpful to have the students start at a fixed point, and with a ruler, find many points that are the same distance from this point. When enough points have been found, it will seem as if all such points lie on a circle. The class could then be asked to find a point in the plane at the given distance from the given point that does not lie on the circle. This, of course, is impossible and illustrates the fundamental property of a locus. Namely, that every point of the locus satisfies the condition and every point that satisfies the condition lies on the locus.

455 We have stated the definition of a locus in terms of a statement and its converse, that is: If a point lies on the locus it satisfies the given condition and, conversely, if the point satisfies the given condition, it lies on the locus. This is the principle used in this chapter to prove locus theorems. However, it is just as valid to prove locus theorems by proving the statement and its opposite. Our definition then is equivalent to: If a point lies on the locus, then it satisfies the given condition, and if a point does not lie on the locus, then it does not satisfy the given condition. In Chapter 16 we will many times prove a statement and its opposite, since this type of proof lends itself to algebraic manipulation. Some reflection will lead us to see that the converse of a statement and the opposite of a statement are logically equivalent. To say that every point which satisfies the condition lies on the locus, is equivalent to saying that every point which does not lie on the locus does not satisfy the condition.

Let us illustrate this point by proving Theorem 14-1 using the opposite of the theorem rather than the converse as the second part of the proof. Let us assume that the first part of the proof (If C is on L, then AC = BC) has been completed, and we proceed to prove the second part. If P is not on L, then PA ≠ PB. Let us take P, any point not on L and show that PA ≠ PB. We will use an indirect method of proof, assuming the conclusion is not true and arrive at a contradiction.



Let P be any point not on L, and let us assume PA = PB. Let M be the mid-point of $\overline{AB}$. If P = M then P is on L, the perpendicular bisector of AB.

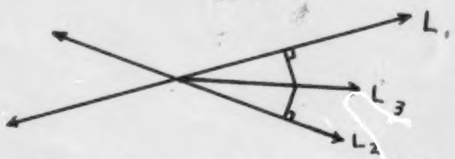
If P ≠ M, then $\triangle PAM \cong \triangle PBM$ by S.S.S. Theorem and $\angle PMA \cong \angle PMB$ by corresponding parts.

Since $\overrightarrow{MA}$ and $\overrightarrow{MB}$ are opposite rays, these angles are supplementary and each of them is a right angle. Since there is a unique perpendicular to $\overleftrightarrow{AB}$ at M, P must lie on the perpendicular bisector of $\overline{AB}$. But this contradicts our hypothesis that P was not on L, hence PA ≠ PB.

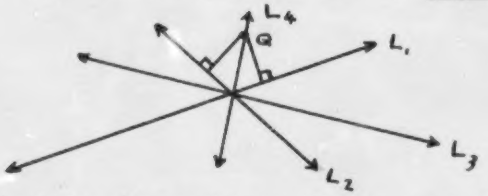
Notice that the indirect proof of the opposite of this theorem is just about a carbon copy of the proof of the converse after it is assumed that the conclusion of the opposite is not true. Some additional information on this appears in SMSG Studies In Mathematics, Vol.II page 2.20 and 2.21.

456 According to the definition, in order to prove a figure is a locus, we must prove a theorem and its converse. If we did not do

this we could not be sure we had the locus. Consider the following example: Find the locus of points equidistant from two intersecting lines. A student might do the problem like this: Having drawn two intersecting lines L_1 and L_2 as below, he might proceed to use the property that the bisector of an angle is equidistant from the sides of the angle and state that L_3, the bisector of one of the angles is the required locus.



His solution, however, is not correct, since he has found only a part of the required locus. If he said that every point on his "locus" was equidistant from the two intersecting lines, he would be correct, but if he were to try and establish that every point that satisfied the given condition was on his "locus," he would readily see his error. For there is a point Q, as in the figure below, that is equidistant from the two lines (L_1 and L_2) that does not lie on his locus. In fact there are many such points which have this property, and we see that the locus is not one of the angle bisectors, but the lines determined by both of them.

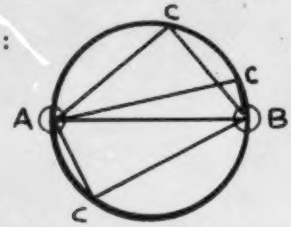


Theorem 14-1 has sometimes been stated, "In a plane two points, each equidistant from the end points of a segment, determine the perpendicular bisector of that segment."

456 In order to illustrate the precision with which we must state locus problems, the following might be presented to the class:

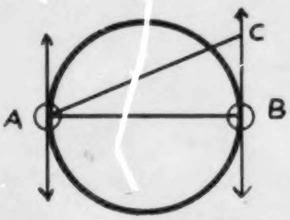
Given two points A and B, what is the locus of points C such that $\triangle ABC$ is a right triangle?

At first thought, one might consider that the angle inscribed in a semi-circle is a right angle and give the following as a picture of the locus:



Note that points A and B are not on the locus.

However, the problem did not say, "What is the locus of point C such that $\triangle ABC$ is a right triangle with right angle at C." The right angle might equally well be at A or at B, and in this case we would have to draw the locus like this:



Note that points A and B are not on the locus.

The locus consists of all points on a circle with diameter $\overline{AB}$ and also all points on the lines perpendicular to the diameter at A and B EXCEPT THE POINTS A AND B.

Theorem 14-3 will be used later to circumscribe a circle about
459 a triangle. The construction is a direct consequence of this theorem. Since this point of concurrency is the center of the circumscribed circle, it is called the "circumcenter".

459 We can answer the question "Why?", as follows. Suppose $L_1 \parallel L_2$. We know $\overleftrightarrow{AB} \perp L_1$ and $\overleftrightarrow{AC} \perp L_2$. Hence $\overleftrightarrow{AB} \perp L_2$. Thus two lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$ as perpendicular to L_2 and must be parallel.

The point of intersection of the altitudes of a triangle is called the "orthocenter".

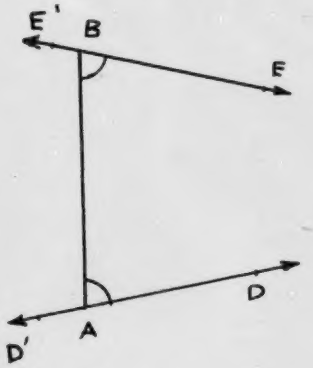
461 Theorem 14-5 will be used to inscribe a circle in a triangle. We can see that the point of intersection is equidistant from the sides of the triangle, and a circle with this point as center and the distance from this point to a side as radius, will have the sides of the triangle as tangents. This point of concurrency is called the "incenter" of the triangle.

460 We have shown in Example 5 of Problem Set 9-7 that the medians of a triangle are concurrent at a point, called the "centroid" of the triangle. It is interesting to note that in a given triangle, the orthocenter, circumcenter and the centroid are collinear. This leads to a very interesting problem. If we draw the segment between the orthocenter and the circumcenter and find its mid-point, then using this point as center and the distance from this point to the mid-points of the sides of the triangle as a radius and draw the circle defined by these conditions, we get what is called the Nine-point Circle. This circle has the following properties: It passes through the mid-points of the sides, it passes through the feet of the three altitudes of the triangle, and it passes through the mid-points of the segments joining the orthocenter (point of intersection of the altitudes) to the vertices. The problem of constructing such a circle is in one of the exercises of this chapter.

In the proof of Theorem 14-4, L_1 is perpendicular to $\overleftrightarrow{DE}$ because $L_1 \perp \overleftrightarrow{BC}$ and $\overleftrightarrow{BC} \parallel \overleftrightarrow{DE}$.

For complete rigor in the proof of this theorem, one should first prove that $\overrightarrow{AD}$ and $\overrightarrow{BE}$ really do intersect. The proof is as follows: Since $m\angle A + m\angle B + m\angle C = 180$, and $m\angle ABE < m\angle B$, and $m\angle BAD < \angle A$, then we have $m\angle ABE + m\angle BAD < 180$.

Now $\overleftrightarrow{BE}$ and $\overleftrightarrow{AD}$ are not parallel, since otherwise we would have $m\angle ABE + m\angle BAD = 180$ and so these lines intersect. (We are using the fact that E and D are on the same side of $\overleftrightarrow{AB}$ to ensure that $\angle ABE$ and $\angle BAD$ are a pair of interior angles on the same side of the transversal $\overleftrightarrow{AB}$).



Let $\overrightarrow{BE'}$ and $\overrightarrow{AD'}$ be the rays opposite to $\overrightarrow{BE}$ and $\overrightarrow{AD}$. Then one of the four cases must hold:

(1) $\overrightarrow{BE'}$ intersects $\overrightarrow{AD'}$. This is impossible since if their point of intersection were T, the triangle TAB would have two angles, the sum of whose measures was more than 180.

(2) $\overrightarrow{BE'}$ intersects $\overrightarrow{AD}$. This is impossible, since the rays lie on opposite sides of $\overleftrightarrow{AB}$.

(3) $\overrightarrow{BE}$ intersects $\overrightarrow{AD'}$. Same reason as (2)

(4) $\overrightarrow{BE}$ intersects $\overrightarrow{AD}$. Being the only possibility left, this must be true.

Notice that we have used no special property of bisectors, merely the fact that $\overrightarrow{BE}$ and $\overrightarrow{AD}$ are in the interiors of $\angle B$ and $\angle A$.

468 From the figure of Theorem 14-6 it appears to be obvious that the two circles intersect, and under the given conditions, we can prove they do intersect. The absence of this theorem in Euclid's Elements is one of the reasons why present-day geometers state that the postulate system of Euclid is incomplete. For a more complete discussion of the need for this theorem see SMSG Studies in Mathematics, Vol. II, pages 1.19 to 1.23.

478 Notice that for every construction, the text gives a proof. When the students do some of the constructions for themselves, some of these should be accompanied by a proof that the construction is correct. A careful analysis of a construction problem will yield a proof with just a little more work.

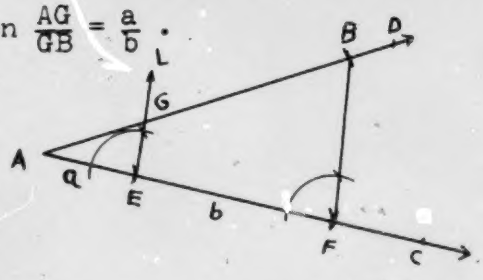
479 Notice how the Two-Circle Theorem is used to establish that the two circles in this problem do actually intersect.

481 Other ways to construct a line parallel to a given line through an external point are (1) construct corresponding angles congruent and (2) construct a line perpendicular to a line through the given point perpendicular to the given line.

481 On the basis of a construction very similar to 14-10 it is possible to divide the length of a given segment in a given ratio. Given a segment $\overline{AB}$, we want to divide $\overline{AB}$ into two segments such that the lengths of these segments will be in some given ratio, say $\frac{a}{b}$

The construction is as follows:

Starting at A draw any ray $\overrightarrow{AD}$, and a ray $\overrightarrow{AC}$ not collinear with ray $\overrightarrow{AD}$. On $\overrightarrow{AD}$ mark off AB and on $\overrightarrow{AC}$ mark off AE = a and EF = b. Draw $\overline{BF}$, and through E construct a line L parallel to $\overleftrightarrow{BF}$ intersecting $\overrightarrow{AD}$ at G. Then $\frac{AG}{GB} = \frac{a}{b}$.



Proof: Since we have in Δ ABF, $\overleftrightarrow{EG}$ parallel to $\overleftrightarrow{BF}$ and intersecting $\overline{AF}$ and $\overline{AB}$, then it follows from Theorem 12-1 that $\frac{AG}{GB} = \frac{AE}{EF}$, hence $\frac{AG}{GB} = \frac{a}{b}$.

489 In the figure at the top of the page, we could also say that ΔABC is circumscribed about circle C_1 and that ΔABC is inscribed in circle C_2.

490 The condition that r should satisfy to insure the intersection of the circular arcs in two points, is that r must be greater than $\frac{1}{2}$ the length of the given segment. In this particular problem, $r > \frac{1}{2}AB$. A value of r that is sure to work is r equal to the length of the given segment; in this problem r = AB will always work.

491 Notice that the construction given for Construction 14.14 works just as well if P is on L.

499 To construct the number in the text requires 15 steps. To verify this, we must know what we mean by a step. A step is one operation of addition, subtraction, multiplication or division. In this example we start with the integers 1,2,3,4,5,7,9,10,17,37, and 47. We construct 5/2 by one step, divide 5 by 2. In like manner, to construct 17/37, 3/4, 7/3, 1/7, 3/5, 9/10 and 37/47 requires one step each; hence to get these numbers we require 8 steps. To construct the numbers $a = 5/2 - 17/37$, $b = 3/4 + 7/3$, $c = 1/7 + 3/5$, and $d = 9/10 - 37/47$ requires 4 steps, two additions and 2 subtractions. We have now used 12 steps and have arrived at four numbers, a, b, c, and d. Now perform two divisions (2 steps) and get the numbers $\frac{a}{b}$ and $\frac{c}{d}$. Now we have used up 14 steps. Finally make one division (1 step) $\frac{a}{b} \div \frac{c}{d}$, and we have now constructed the number given in the text in 15 steps. The number is rational, $\frac{168,063}{1,672,918}$

500 In the figure for the trisection problem, it is interesting to see that as m $\angle$ ABC increases the marked point P moves in a very limited range on the ray opposite to $\overrightarrow{BA}$. This range is $\sqrt{2}\, r \leqslant PB < 2r$.

If C coincides with A then PB = 2r and we do not have an angle to trisect. As $m\angle ABC$ increases C and Q approach coincidence. When they coincide the ruler cuts the circle in only one point Q, and $\overline{BQ} \perp \overline{PQ}$ and $BP = \sqrt{2}r$. The largest angle we can trisect by this method is 135°. The trisection of any obtuse angle can be reduced to the trisection of an acute angle.

Chapter 15

503 Notice that the definition given in the text on this page includes polygons, which when taken with their interiors, do not form convex sets. We will find that the text will consider only convex polygons. Thus far no mention has been made of "regular" polygons. These will be discussed in Section 15-2.

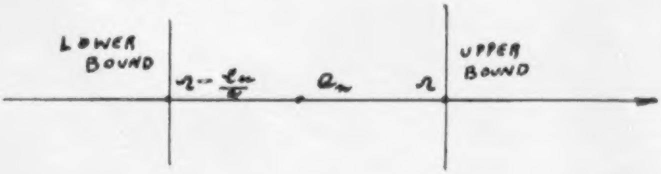
The apothem of a regular polygon can also be described as (1) the perpendicular from the center to a side, or (2) the radius of the inscribed circle of the polygon.

512 Beginning in Section 15-3 the text introduces the notion of a limit. It is not intended that the students be given a formal treatment of limits, but rather that they develop an intuitive idea of what a limit is. It is rather easy to visualize what happens to an inscribed regular polygon as we let the number of sides increase indefinitely. In discussing limits with a high school class, a teacher once sought to develop the topic in this manner. She asked, "If a boy and a girl were to each sit at opposite ends of a bench and each were to continue moving toward the other so that with each move they halved the distance between, would they ever meet?" The reply was, "No, but for all practical purposes they would get close enough!" Humorous though this may seem, this is the idea we are trying to get across. The respondent, whomever he or she was, actually had a pretty good intuitive idea of what a limit was.

It will be noticed that in the discussions concerning limits, no mention of "infinity" has been made, nor has any symbol been introduced. The concept of a limit does not involve any notion of infinity. While the word and the symbol (∞) for it are convenient in certain branches of higher mathematics, they should be avoided in introductory discussions where they are neither useful nor enlightening.

514 Some explanation of why $\frac{P_n}{r} = \frac{P_n'}{r'}$ is true after we know $\frac{e_n}{r} = \frac{e_n'}{r'}$ might be helpful. Since we are dealing with regular polygons, each side of which is congruent to every other side, by adding the ratios of the sides to the radii $\left(\frac{e_1}{r} + \frac{e_2}{r} + \frac{e_3}{r} + \ldots + \frac{e_n}{r}\right)$ we see that the ratio of this sum is the ratio of the perimeter to the radius. Another way to see this is to take the equation $\frac{e_n}{r} = \frac{e_n'}{r'}$ and form the equivalent equation $\frac{ne_n}{r} = \frac{ne_n'}{r'}$, then $\frac{P_n}{r} = \frac{P_n'}{r'}$ since $P_n = ne_n$.

517 The inequality $r > a_n > r - \frac{e_n}{2}$ can be expressed graphically in the following manner, and we can then visualize what happens to a_n as $e_n \rightarrow 0$. Since we know from the inequality, that a_n lies between $r - \frac{e_n}{2}$ and r, for all n, we can illustrate this below:



From the proof we see that the value of r is fixed, and as $e_n \rightarrow 0$, the lower bound approaches the upper bound. Thus a_n, which is "squeezed" between the two bounds, must also approach the fixed upper bound r. We could also consider the problem in this manner. We see that the length of the interval in the figure above is $\frac{e_n}{2}$, and as this quantity approaches 0 then a_n approaches the fixed value r as a limit.

517 Notice that we have not defined the area of a circle as the limit of the areas of the inscribed regular polygons. Rather, we are assuming that a circle has an area, and that (1) its area is greater than the area of any polygonal region that lies in the circle, and (2) its area is less than the area of any polygonal region which contains the circle. Hence once we have shown that $A_n \rightarrow \pi r^2$, we cannot state the formula for the area of a circle. Since we have been working only in the interior of the circle, we only know that the area of the circle is greater than the area of the inscribed polygon for all m, therefore $A \geq \pi r^2$. To complete the argument and show $A = \pi r^2$ we must, in a sense, "bracket" the circle with a circumscribed polygon. Then the area of the circle will be less than the area of the circumscribed polygon for all m. Once we have done this we can show that $\pi r^2 \geq A$. Then $\pi r^2 \geq A \geq \pi r^2$. Hence we finally arrive at $A = \pi r^2$.

527 Note that in our use of the word "cross-section", the intersecting plane must be parallel to the base. It is possible to have sections formed by passing a plane through the prism at any angle, and the section thus formed would not possess the same properties as a cross-section. It should be noted that a cross-section of a prism is a convex polygonal region.

In Theorem 15-3, the text states that the cross-sections of a triangular prism are congruent to the base. Up to this point no mention has been made of the congruency of triangular regions, but only of the congruency of triangles. It is intuitively apparent that if two triangles are congruent, then their associated regions would also be congruent. We will not speak of the congruency of any polygonal regions other than triangular regions, since any polygonal region can always be divided into triangular regions.

530 In part (1), $\overline{A'P'} \parallel \overline{AP}$ for the following reason: Points V, A, and P determine a plane which intersects the two parallel planes (E_2 and E_1) in $\overleftrightarrow{A'P'}$ and $\overleftrightarrow{AP}$. By Theorem 10-1, $\overleftrightarrow{A'P'} \parallel \overleftrightarrow{AP}$. $\triangle A'P'V \sim \triangle APV$ by Corollary 12-3-1 (A.A. Corollary) as follows: $\angle VP'A'$ and $\angle VPA$ are right angles since a line perpendicular to one of two parallel planes is perpendicular to the other, and since $\overleftrightarrow{A'P'} \parallel \overleftrightarrow{AP}$, then by corresponding angles $\angle VA'P' \cong \angle VAP$, hence the triangles are similar.

The triangles in (2) can be proved similar by either the AAA Similarity Theorem or the AA Corollary. Once we have proved the triangles similar, we can establish from the equality of the 9 ratios that:

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC}$$, and by the SSS Similarity Theorem,

$$\triangle A'B'C' \sim \triangle ABC.$$

533 Since by Theorem 15-4, $\frac{A_1'}{A_1} = \frac{k^2}{h^2}$, by algebra we get $A_1' = \frac{k^2}{h^2} A_1$ etc.

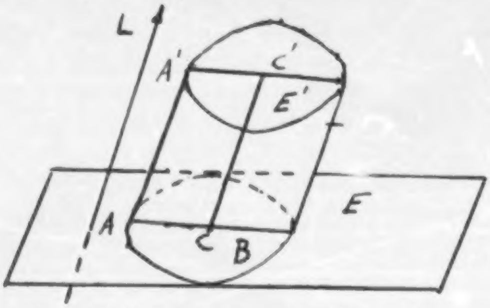
534 Notice that, as was done with areas, the text postulates the formula for the volume of one solid and proceeds to prove the remaining formulas for the volumes of prisms, pyramids, cones, cylinders and spheres, in that order.

Cavalieri's Principle is an extremely powerful postulate. This postulate can be proved by methods resting on the theory of limits as developed in the integral calculus. This postulate will be used throughout the chapter to prove theorems concerning the volumes of solids.

538 In order to help the students visualize how three triangular pyramids are formed by cutting a triangular prism, some visual aid could be used. Disected solids can be purchased from an equipment supply company, they can be made, or they can be formed by cutting up a bar of soap.

The three pyramids are formed by cutting the triangular prism by the planes through points S,P,R and the points S,P,U.

547 Proof that every cross-section of a cylinder is a circle plus its interior with the same radius as the base:



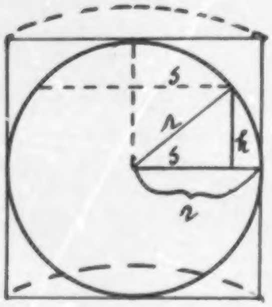
Let E and E' be two parallel planes. Let L be any line intersecting E and E', and let B be a circle in E. Then, by the definition of a cylinder, we can form a cylinder having B as its base and $\overleftrightarrow{AA'} \parallel L$. Through C, the center of B, draw a line parallel to L, intersecting E' at C'. Since $\overleftrightarrow{AA'} \parallel L$ and $\overleftrightarrow{CC'} \parallel L$, then by Corollary 10-4-2 $\overleftrightarrow{AA'} \parallel \overleftrightarrow{CC'}$, and these two lines determine a plane which intersects E' and E in $\overleftrightarrow{A'C'}$ and $\overleftrightarrow{AC}$ respectively. Since $E' \parallel E$, $\overleftrightarrow{A'C'} \parallel \overleftrightarrow{AC}$. Therefore ACC'A' is a parallelogram and A'C' = AC. Since A was any point on B, every point on the boundary of the cross-section is at a distance AC from C'. Hence the cross-section must be a circle plus its interior with the same radius as the base of the cylinder.

547 Since we know how to find the volume of the cylinder, we can find volume of the sphere by subtracting the volume of the solid F

from the volume of the cylinder. That is:

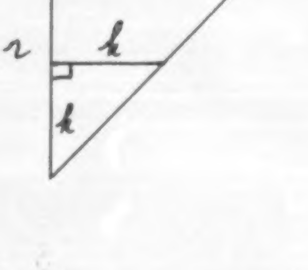
$$V = V_1 - V_2.$$

549 The equation, $s^2 + k^2 = r^2$ is obtained from the figure on page 547 as follows: We see that the radius of the cross-section of the cylinder is r, and this is also the radius of the sphere. If we pass a plane vertically through the center of the solid we get the figure below:



We see now that in the right triangle formed, $s^2 + k^2 = r^2$, by the Pythagorean Theorem.

549 The answer to the "Why?" is as follows:
The altitude of each of the two cones is r. Since the radius of each of the circles which are the bases of the right cylinder is r, we have an isosceles right triangle formed by the altitude, the radius, and a segment on the surface of the cone joining the vertex to a point on the base. Any line parallel to the radius, intersecting the other two sides, will form a triangle similar to the original one.



Hence the cross-section of the cone at a distance k from the vertex (which is also the center of the sphere) will be a circle with radius k, and its area is then πk^2 , which is also the area of a cross-section (an annulus) of the solid F.

Chapter 16

This chapter on coordinate geometry has been included in the text for various reasons. Since Chapter 4 the student has been acquainted with a coordinate system on a line and the transition to coordinate geometry in the plane should be as simple as possible for him. Many properties of synthetic geometry can be established and proved more easily by analytical methods than by synthetic methods. This is not to say that because we can do something more easily by one method than another, that we should abandon one completely in favor of the easier method. To do so would rob the student of the appreciation of the power of a synthetic deductive system. On the other hand, there are many exercises, with relatively easy synthetic proofs, that become extremely difficult when done analytically. The most reasonable course of action then, appears to be to develop an appreciation for, and the ability to use, synthetic methods, and then take from the analytic geometry that which will enable the student to realize that there are different methods of attacking the same problem. It is not intended that this chapter be considered a complete introduction to analytic geometry. We have taken from the subject those tools which will enable us to do the job, namely to work some exercises that can be solved efficiently by analytical methods.

555 The history of geometry, like the history of all of mathematics, is a fascinating story. When one knows the history of a subject, he can better appreciate the years of development necessary to put it into the form we know it today. Since the development of analytic geometry was a major break-through in mathematical thought at the

time Descartes discovered it, the students might be interested in the history of its development and discovery, just as they might be interested in the history of synthetic geometry.

556 Notice that we now set up a coordinate system on each of two perpendicular lines, rather than on only one line, as we did in Chapter 4. This enables us to find the coordinates of its projections on the two lines. We write these coordinates as an ordered pair (x,y).

557 We again have a one-to-one correspondence, this time between ordered pairs of real numbers and points in a plane. To each ordered pair of real numbers there corresponds one and only one point in the plane, and to each point in the plane there corresponds one and only one ordered pair of real numbers.

Some of the material on these introductory pages may be in the nature of a review. In this case, the pupils should be moved along as rapidly through this material as their background will permit.

560 From this point on, the x-axis and y-axis will be labeled "x" and "y" respectively. Note that these are lower case letters and not the capitals used in previous figures.

563 When we define the slope of a line to be the quotient of the difference between pairs of coordinates, there is no need to introduce the notion of directed distance, but it is absolutely necessary to put the coordinates of the two points (x_1, y_1) and (x_2, y_2) in the proper position in the formula. That is $m = \frac{y_2 - y_1}{x_2 - x_1}$ cannot be used as $m = \frac{y_2 - y_1}{x_1 - x_2}$ although $m = \frac{y_1 - y_2}{x_1 - x_2}$ is also correct. Notice that in finding the slope of segment AB it doesn't matter which point is labeled P_1 and which one is labeled P_2.

566 It is important to note here that RP_2 and P_1R are positive numbers and we have to prefix the minus sign to the fraction $\frac{RP_2}{P_1R}$ if the slope is negative. It is also clear that the formula for the slope of a line will give the slope m as positive or negative without prefixing any minus sign.

For the case (1) if $m > 0$, then $m = \frac{RP_2}{P_1R}$, $RP_2 = y_2 - y_1$ and $P_1R = x_1 - x_2$.

For the case (2) If $m<0$, then $m = -\frac{RP_2}{P_1R}$, $RP_2 = y_2 - y_1$ and $P_1R = x_1 - x_2$.

Case (2) may look wrong, but if $m = -\frac{y_2 - y_1}{x_1 - x_2}$ then $m = \frac{y_2 - y_1}{x_2 - x_1}$ which is the formula for the slope of a line. The essential thing to make sure of is that the students understand that either $y_2 - y_1$ or $y_1 - y_2$ is a positive number, so RP_2 and P_1R can always be calculated by choosing the proper order for the difference involved.

570 thru 574 The information concerning slopes of parallel and perpendicular lines constitutes a very important tool in the solving of geometric problems analytically. For instance, if a student were asked to show that two lines were parallel, all he would have to do is show that their slopes were equal; to show that a pair of lines were perpendicular would require that he establish the slopes to be negative reciprocals of each other. In the proof that parallel lines have the same slope and conversely, the figure showing only positive slopes was used in the proof, but the proof would go through using negative slopes as well.

573 Here is a proof that $\triangle PQR \cong \triangle Q'PR'$.

(1) $\overline{PQ} \cong \overline{Q'P}$ (1) By hypothesis

(2) $\overline{Q'R'} \perp \overline{R'P}$ $\overline{QR} \perp \overline{RP}$	(2) By hypothesis
(3) $\overline{PQ} \perp \overline{PQ'}$	(3) By hypothesis
(4) $m\angle R'PQ' + m\angle Q'PQ + m\angle RPQ = 180$	(4) $\overrightarrow{PR}$ and $\overrightarrow{PR'}$ are opposite rays and Postulate 14
(5) $m\angle R'PQ' + m\angle RPQ = 90$	(5) Subtraction
(6) $m\angle R'PQ' + m\angle R'Q'P = 90$	(6) Acute angles of a right triangle are complementary.
(7) $\angle RPQ \cong \angle R'Q'P$	(7) Complements of the same angle, or congruent angles, are congruent.
(8) $\angle Q'PR' \cong \angle RQP$	(8) Corollary 9-12-1
(9) $\triangle PR'Q' \cong \triangle QRP$	(9) A.S.A. Theorem

We next need to prove that if $m' = -\frac{1}{m}$ then $L_1 \perp L_2$. We will do this by using the S.A.S. Postulate.

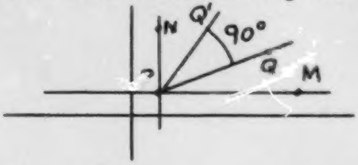
By hypothesis make $R'P = RQ$, then complete the construction of right triangle $Q'PR'$, and $\angle R$ and $\angle R'$ are right angles by construction. Now, $m = \frac{RQ}{PR}$ and $m' = -\frac{R'Q'}{R'P}$. Then by hypothesis: $m' = -\frac{1}{m}$, therefore $-\frac{R'Q'}{R'P} = -\frac{PR}{RQ}$. Since by hypothesis the denominators $R'P$ and RQ are equal, it follows that $R'Q' = PR$. In $\triangle PQR$ and $\triangle Q'PR'$ we have: $m\angle R = m\angle R'$ since they are both right angles; $PR' = QR$ by hypothesis and from above $R'Q' = RP$.

Therefore $\triangle PQR \cong \triangle Q'PR'$ by the S.A.S. Postulate.

Now let $m\angle RPQ = \alpha$. Then $m\angle R'QP' = \alpha$, since corresponding parts of congruent triangles are congruent. From the addition postulate we now get $m\angle Q'PR = m\angle Q'PQ + \alpha$. And since $\angle Q'PR$ is an exterior angle of the triangle we get $m\angle Q'PR = 90 + \alpha$. Therefore $m\angle Q'PQ + \alpha = 90 + \alpha$ and by algebra $m\angle Q'PQ = 90$.

Hence $L_1 \perp L_2$ by the definition of perpendicular.

Points Q and Q' were chosen so that $\overleftrightarrow{PQ}$ had positive slope and $\overline{PQ'}$ had negative slope. This is possible for if both had positive or negative slope the picture would be



$\overleftrightarrow{PN} \parallel$ y-axis and $\overleftrightarrow{PM} \parallel$ x-axis and by angle addition postulate $m\angle NPM = 90^\circ = m\angle MPQ + 90^\circ + m\angle QPN$ which is impossible.

That $\overline{AB} \cong \overline{CB}$ because the length of each is $\sqrt{c^2 + 2b^2}$, is just as valid as saying that they are congruent because each of their lengths is 6. Notice that it would have been impossible for us to have developed the distance formula without the Pythagorean Theorem, which in turn rested upon the theory of areas, parallels, and congruence. It might be instructive with a good class to have them derive the distance formula with P_1 and P_2 in various positions in the plane.

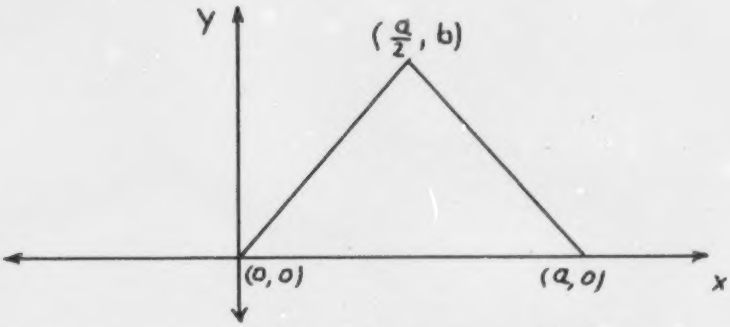
In working with the distance formula, it does not matter in which order we take P_1 and P_2, inasmuch as we will be squaring the difference between coordinates. The distance formula holds even when the segment P_1P_2 is horizontal or vertical.

576 The symbol $P_1P_2{}^2$ means the same thing as $(P_1P_2)^2$. If the notation used in the text confuses the students, then use the parentheses.

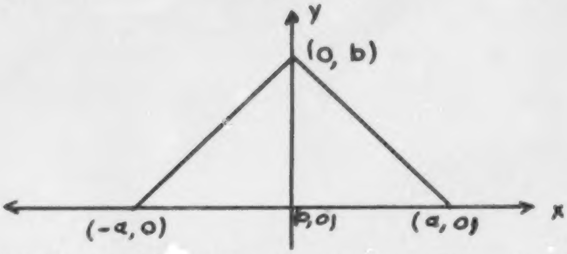
580 The mid-point formula, too, will prove to be a very useful tool in the work which follows. This will be particularly true when we are speaking of medians of triangles, for example. If we know the coordinates of the vertices of a triangle, and apply the definition of a median (that it bisects the side to which it is drawn), then we can identify by means of the coordinates of the vertices the

point (or points) which a median (or medians) intersect the opposite side. This would then give us the coordinates of its end-points and enable us to find the length and slope of the median, if it were desired, and it may be. The proof of the mid-point formula does not hold for horizontal and vertical segments, but the formula does hold.

58[4] Although we may chose coordinates of some points in any manner we wish, and we may place our axes in any manner we desire in relation to the figure there are some limits on what we can do with coordinates. For instance, if we are given an isosceles triangle, we may place the axes wherever we wish, then use the properties of an isosceles triangle to determine the coordinates of the vertices. Suppose we place it like this:

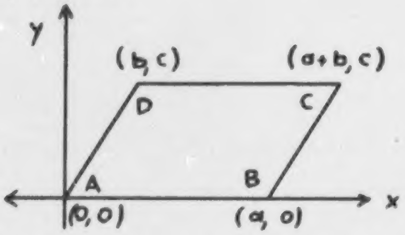


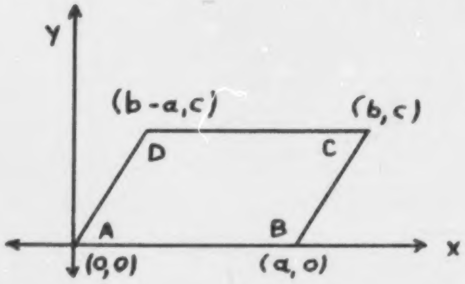
The student should be permitted to draw upon his information of synthetic geometry to the extent that he knows that the altitude to the base of an isosceles triangle bisects the base. Hence, its x-coordinate should be $\frac{1}{2}$ the x-coordinate of the vertex of the base that is not at the origin. On the other hand we know nothing about the y-coordinate of the vertex, and hence it is not dependent upon the coordinates of the other vertices. If we place the axes like this: (such that the vertex is on the y-axis.)



Then, since the altitude bisects the base, the lengths of the segments on either side of the altitude must be equal, and therefore should be indicated by some type of relationship such as (a,0) and (-a, 0). For parallelograms, we find that generally three vertices may be labeled arbitrarily, but the coordinates of the fourth are fixed, *provided we know that the figure given is a parallelogram*.

Again, there is more than one way in which we may label a figure such as a parallelogram. In the figure on the left below the coordinates of points A, B, and D had no special restrictions placed upon them (unless they were on an axis). The coordinate of C could be determined in terms of the coordinates of the other three points. In the figure on the right below, A,B, and C were chosen first, and notice how the coordinates of D are given in terms of the other coordinates.



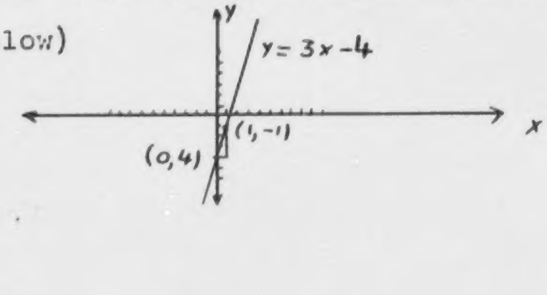


One word of CAUTION. The above discussion is based upon the fact that such things as isosceles triangles or parallelograms are *given* in the problem. If the problem were such that we were to prove a quadrilateral to be a parallelogram or a triangle to be isosceles, then we cannot assume such properties to be true, and must establish, as part of the exercise, that the coordinates bear some relation to one another.

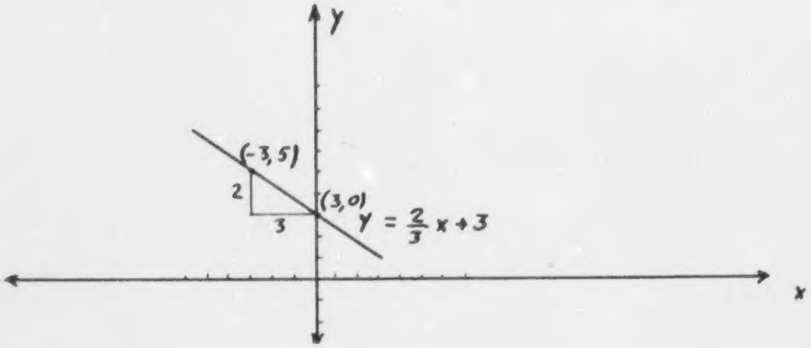
602 thru 604 The material in Section 16-11 may also have been previously covered in a first year algebra course. If this is the case, do not spend any more time than is necessary on this section.

603 You will note that in the discussion on this page, it is necessary for us to find an additional point in order to plot the graph of the equation.

We may do this in two ways. The first would be to assign to x any arbitrary value, substitute this value in the given equation and compute the corresponding value of y (or we could assign a value to y and compute x). The second method depends upon the discussion on page 16.10 and 16.11. Since we know how a line with a positive or negative slope will fall, and we also know, from page 16.11, that if a line has a positive slope then $m = \frac{RP_2}{P_1R}$ and if its slope is negative, $m = -\frac{RP_2}{P_1R}$, then given one point on the graph and the slope we can find a second point by counting the units in the legs of the right triangle. Consider the following example from page 16.41: Draw the graph of $y = 3x - 4$. We see immediately that the point (0,4) lies on this graph and that its slope is 3. Since the slope is positive, the graph will rise to the right. Hence we can find a second point by starting at (0,4) and counting 1 unit to the right and three units up which brings us to (1,-1). We can check to see if we are correct by applying the slope formula to this point pair and see that the slope of the segment between these two points is truly 3. (See figure below)



Let us consider one more case, namely when the slope of the given equation is negative. Draw the graph of the equation $y = -\frac{2}{3}x + 3$. We see that the point (0,3) lies on the graph, and to locate a second point by this method, we must realize that we will be working with a slope of $-\frac{2}{3}$. The graph, then, will rise to the left and we can locate a second point by counting 3 units to the left from (0,3) and 2 units up, as in the figure below.



605 thru 608 The material on these pages might be new to the class. The subject matter is not difficult and the proofs are easy to follow.

616 Notice how the method of completing the square is needed to transform an equation of a circle into the standard form. This was mentioned in the manual referring to page 16 of Chapter 2.

GEOMETRY

Teacher's Manual

Part II

Solutions to the Problems

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Solutions of Problems

Problem Set 8-2.

1. a. F. b. F. c. F. d. T. e. T. f. F. g. T. h. T.

2. The line of intersection is perpendicular to the floor. Many, in fact, every line going through the point at which L intersects the floor will be perpendicular to L. No. It is perpendicular only to lines of the floor that contain the point of intersection of L and the floor.

3. AR > RB. $m\angle B \quad m\angle A$ $(m\angle RBA = 90)$.

4. a. 3. Planes ABF, RHB, and AHR.

   b. $\overline{HB} \perp \overline{RH}$. (Given.) $\overline{HB} \perp \overline{AF}$. (The median to the base of an isosceles triangle is perpendicular to the base.) Therefore, $\overline{HB} \perp$ plane AHRF.

5. a. 3. Plane ABF. Plane RHB. Plane RHF. (Also contains A.)

   b. $\overline{AF} \perp \overline{RH}$

      so, $\overline{AF} \perp$ plane RHB.

      $\overline{AF} \perp \overline{BH}$

6. a. 3. The sides of the square all lie in a plane. $\overline{AB}$ and $\overline{FB}$ determine another plane, and $\overline{AB}$ and $\overline{BH}$ determine a third.

   b. We know $\overline{BH} \perp \overline{HR}$, $\overline{HR} \perp \overline{RF}$, $\overline{RF} \perp \overline{FB}$, $\overline{BF} \perp \overline{BH}$ (from the square) and $\overline{BF} \perp \overline{AB}$ (Given.). In the last two of these we have one line, $\overleftrightarrow{FB}$, to two other lines at their point of intersection so we know that $\overleftrightarrow{FB} \perp$ to the plane ABH. It is also true that $\overline{RH} \perp$ plane ABH, but the student probably cannot prove this now.

7. Plane $F \perp \overleftrightarrow{WX}$, and so, $\overleftrightarrow{WX} \perp$ plane F. $\overleftrightarrow{WX}$ is, therefore, perpendicular to $\overleftrightarrow{QR}$. $\overleftrightarrow{AB} \perp \overleftrightarrow{RQ}$ (Given). We now have $\overleftrightarrow{RQ}$ perpendicular to both $\overleftrightarrow{WX}$ and $\overleftrightarrow{AB}$ at Q, their point of intersection, so $\overleftrightarrow{RQ} \perp E$.

8. Proof:

Statements	Reasons
1. $\overline{FB} \perp$ plane p.	1. Given
2. $\overline{FB} \perp \overline{AB}$ $\overline{FB} \perp \overline{RB}$.	2. Def. of line perpendicular to a plane.
3. $m\angle FBA = m\angle FBR = 90$.	3. Definition of perpendicular lines.
4. $BR = BA$.	4. Given.
5. $FB = FB$.	5. Identity.
6. $\triangle ABF \cong \triangle RBF$.	6. S.A.S.

9. $\triangle FQA \cong \triangle FQB$ (S.A.S.) and so, $FA = FB$. $FR > FA$. Let X be the intersection of $\overline{QR}$ and the circle. Then $\triangle FQX \cong \triangle FQA$ and $FX = FA$. In $\triangle FQR$, since $FQ \perp QR$, $m\angle FQR$ $m\angle R$ and $m\angle FXR > m\angle FQR$, so

$m\angle FXR > m\angle R$. $\therefore$ in $\triangle FXR$, $\overline{FR}$ and $\overline{FX}$ are opposite incongruent $\angle$s, with $\angle FXR$ the larger. Thus $FR > FX$, i.e., $FR > FA$.

10. Connect A with X, the point of $\overrightarrow{BF}$ such that $BX = BH$. Then $\triangle ABH \cong \triangle ABX$ (S.A.S.) and $AX = AH$. Since $\overline{AB} \perp \overline{BF}$, $m\angle ABF$ $m\angle F$, and since $\angle AXF$ is an exterior angle of $\triangle ABX$, $m\angle AXF > m\angle ABX > m\angle F$. Then $AF > AX$ and, substituting, we have $AF > AH$.

11. On ray $\overrightarrow{QR}$ opposite ray $\overrightarrow{QB}$ let R be the point such that $QR + QB$. Then $\triangle PQR \cong \triangle PQB$ by S.A.S. $\therefore PR = PB$. $\overline{AP} \perp \overline{PR}$, since $\overline{AP} \perp$ plane PBC and R is in plane PBC. Therefore, $\triangle APR \cong \triangle APB$ (S.A.S.) $\therefore \overline{AQ} \perp \overline{RB}$, since the median to the base of an isosceles $\triangle$ is perpendicular to the base.

12. Proof:

1. A - H is a cube.	1. Given
2. $AT = TF$. $AB = BF$.	2. A - H is a cube.
3. $BR = BL$.	3. Given.
4. $AR = FL$.	4. Subtracting Step 3 from Step 2.
5. $\triangle ATR \cong \triangle FTL$.	5. S.A.S.
6. $TR = TL$.	6. Definition of congruence.
7. $\overline{KT} \perp \overline{AT}$. $\overline{KT} \perp \overline{FT}$.	7. A - H is a cube.

8. $KT \perp$ plane ABFT and hence $\overline{KT} \perp \overline{TR}$ and $\overline{KT} \perp \overline{TL}$.	8. Definition of a line perpendicular to a plane.
9. $\Delta KTR \cong \Delta KTL$.	9. S.A.S.

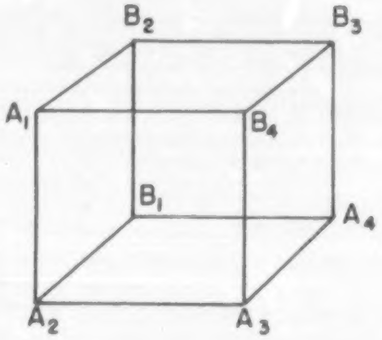
13. $\overline{BW}$. $\overline{BK}$. $\overline{BR}$. 90. $\angle BKF$.

Not necessarily.

14. Yes. Yes. No.

15. One.

16.



17. Let X be any point of L Since P and Q are each equidistant from A and B so is X by Theorem 1. Similarly X is equidistant from B and C. Thus

$$XA = XB = XC$$

that is X is equidistant from A, B, C.

18. $\Delta XAM \cong \Delta XBM$ by S.A.S.

Hence $XA = XB$. Similarly we show

$$XB = XC$$

Thus X is equidistant from A, B, C.

19. X is equidistant from A and B and so is in the perpendicular bisector of $\overline{AB}$. Similarly X is in the perpendicular bisector of $\overline{BC}$. Hence X is in the line of intersection of these planes. Similarly using Problem 2 we see that L lies in each of these planes. Thus L is their intersection and must contain X.

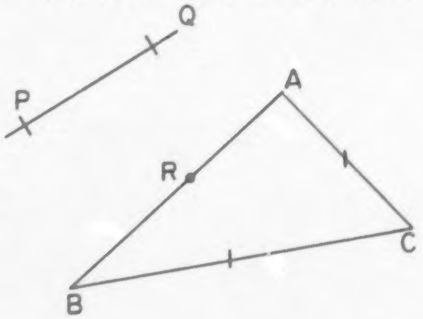
20. Let Q be the intersection of L and plane E. Then $\triangle\ PQ \cong \triangle PQB$ by Hypotenuse-Leg Theorem

$$\therefore QA = QB$$

Similarly $QB = QC$. Thus Q is equidistant from A, B, C and so is every point of L by Problem 17.

21. Suppose in plane E two points P and Q are equidistant from A, B, C which are non-collinear. Let R be the midpoint of AB. Then P,Q,R are equidistant from A and B. Suppose R is not on line PQ. It follows by Theorem 2 that all points of plane E are equidistant from A and B. Thus A would be equidistant from A and B which is absurd. Consequently, our supposition is false and R is on $\overleftrightarrow{PQ}$, that is $\overleftrightarrow{PQ}$ bisects $\overline{AB}$. Similarly it bisects $\overline{BC}$ and $\overline{AC}$. This is absurd and our original supposition is false.

Another solution. Let $L \perp E$ at P. By Problem 3, Q is in L since Q is equidistant from A, B. C. Thus L = PQ which lies in E. This is impossible since $L \perp E$.



Problem Set 9-1.

1. a. No. b. No.

2. Show that (1) they do not intersect, or (2) they are both perpendicular to a third line or (3) they form alternate interior angles with a transversal.

   (Note: the third method includes the second as a special case.)

3. No.

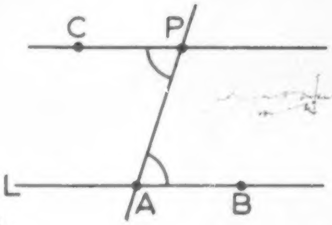
4. Not necessarily.

5. a. No, the 80° angles are not alternate interior angles, and the alternate interior angles are not equal.

   b. Two sizes: 80° and 100°.

6. $L_1 \parallel L_2$, $M_1 \parallel M_2$.

7. Select any two points A,B on L. Draw $\overleftrightarrow{PA}$. Draw $\angle CPA \cong \angle BAP$ so that C and B are on opposite sides of $\overleftrightarrow{PA}$. Then $\overleftrightarrow{CP} \parallel L$ by theorem 9-4.



8. a. Yes. b. No. c. Yes. d. Yes. e. Yes, since a line containing the center of the earth is perpendicular to certain other lines containing the center.

   f. No. g. Yes. h. Yes.

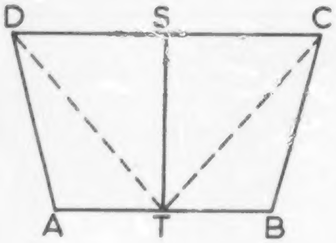
9. Yes. (Such lines are called skew lines.)

10. Proof: $\triangle ABD \cong \triangle BAC$ by S.A.S. Then $DB = CA$. Then $\triangle DCB \cong \triangle CDA$ by S.S.S. and $m \angle BCD \cong m \angle ADC$. (It is not possible to prove that $\angle BCD$ and $\angle ADC$ must be right angles. Attempts to do so suggest the need for some further postulate.)

11. Proof: $\triangle APR \cong \triangle PBQ \cong \triangle RQC \cong \triangle QRP$ by S.S.S. By corresponding parts m $\angle a$ = m $\angle b$ = m $\angle B$ and m $\angle c$ = m $\angle C$. Since the sum of the measures of angles a, b and c is 180 by postulates 13 and 14, the sum of the measures of angles A, B and C is 180.

12. Proof: $\triangle PAR \cong \triangle QAR$ by S.A.S. Then $\angle ARP \cong \angle ARQ$ and $\overleftrightarrow{AR} \perp \overleftrightarrow{PQ}$. By a similar proof using triangles ABD and ACD, $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$. Then $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$ by theorem 1. (Note: a proof based on isosceles triangles without drawing AD is also possible.)

13.



1. $\triangle DAT \cong \triangle CBT$.	1. S.A.S.
2. DT = CT.	2. Corresponding parts.
3. m $\angle DTA$ = m $\angle CTB$.	3. Corresponding parts.
4. $\triangle DST \cong \triangle CST$.	4. S.S.S.
5. m $\angle DTS$ = m $\angle CTS$.	5. Corresponding parts.
6. m $\angle STA$ = m $\angle STB$.	6. Addition.
7. $\overleftrightarrow{ST} \perp \overleftrightarrow{AB}$.	7. Definition of perpendicular lines.
8. m $\angle TSD$ = m $\angle TSC$.	8. Corresponding parts.
9. $\overleftrightarrow{ST} \perp \overleftrightarrow{CD}$.	9. Definition of perpendicular lines.
10. DC $\parallel$ AB.	10. Two lines which are perpendicular to the same line are parallel.

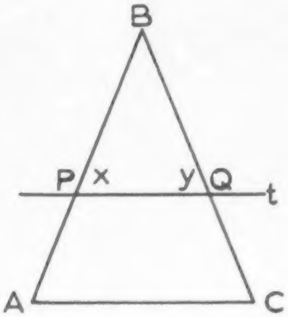
Problem Set 9-3.

1. Proof:

1. $m\angle A = m\angle B = m\angle C = 90$.	1. Given.
2. $\overline{AD} \parallel \overline{CB}$.	2. In a plane two perpendiculars to the same line are parallel.
3. $m\angle C = m\angle D = 90$.	3. Two parallels form supplementary interior angles on the same side of a transversal.

2. Given: Isosceles $\triangle ABC$ with $AB = BC$ and $t \parallel \overline{AC}$ and intersecting $\overline{AB}$ and $\overline{BC}$ at P and Q.

Prove: $\triangle PBQ$ is isosceles.



Proof:

1. $\angle A \cong \angle C$.	1. Base angles of an isosceles triangle are congruent.
2. $t \parallel \overline{AC}$.	2. Given.
3. $\angle x \cong \angle A$ and $\angle y \cong \angle C$ so that $\angle x \cong \angle y$.	3. Parallel lines form congruent corresponding angles.
4. $\overline{PB} \cong \overline{BQ}$, or $\triangle PBQ$ is isosceles.	4. Sides opposite congruent angles of a triangle are congruent.

3. Proof:

$\angle PQT \cong \angle S$ since they are corresponding angles formed by a transversal of two parallels.

$\angle RTS \cong \angle S$ because they are base angles of an isosceles triangle.

From these two statements $\angle PQT \cong \angle RTS$. Then $\overline{PQ} \cong \overline{QT}$ because they are opposite congruent angles of $\triangle PTQ$.

4. a. Proof:

Suppose M does not intersect L_2. Then, by definition, $M \parallel L_2$. But L_1 is given $\parallel L_2$. Hence there are through P two parallels to L_2-- an impossibility by the Parallel Postulate. The assumption that M does not intersect L_2 is therefore false, so that M does intersect L_2.

b. Proof:

Suppose $L_1 \parallel L_2$. $R \parallel L_2$ by the given information. Also by the given information both L_1 and R contain P. Since there cannot be two parallels to a line through a point, the assumption $L_1 \parallel L_2$ is false, and L_1 intersects L_2.

5. a. Proof:

$\angle RBC \cong \angle RYZ$ and $\angle RBA \cong \angle RYX$ because they are corresponding angles. By subtraction of the measures of these angles $m \angle ABC = m \angle XYZ$. (If one of the angles had sides extending in opposite directions from the other, its congruent vertical angles would have sides extending in the same direction and the preceding proof could be applied.)

5. b. Proof:

By drawing line $\overleftrightarrow{YX}$ form ∠PYX with sides extending in the same direction as those of ∠ABC.

Then, from part (a), m ∠PYX = m ∠ABC. But m ∠PYZ + m ∠XYZ = 180, and therefore m ∠ABC + m ∠XYZ = 180.

6. If the sides of one angle are perpendicular respectively to the sides of another angle, then the angles are either congruent or supplementary.

7. Proof:

Draw a transversal $\overleftrightarrow{PQ}$ of L_1 and M and also of L_2 and M forming angles a, b and c as shown. If $L_2 \parallel M$, then ∠b ≅ ∠c; and since $L_1 \parallel M$, ∠a ≅ ∠c by Theorem 9-7. Therefore, ∠a ≅ ∠b. But then $L_1 = L_2$ by the Angle Measure Postulate, so there cannot be a second parallel to M through Q.

8. Proof:

Consider a line t perpendicular to M from P. By Theorem 9-6, t ⊥ L_1. Assume L_2 parallel to M. Then t ⊥ L_2. Since L_1 and L_2 cannot both be perpendicular to t at P, L_2 cannot be parallel to M as was assumed.

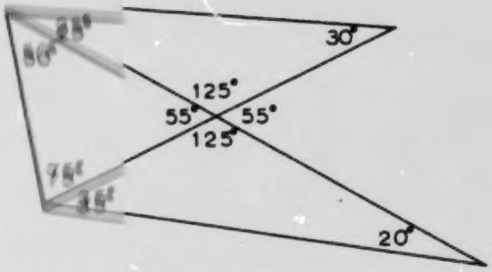
Problem Set 9-4.

1. a. 85. b. 1. c. 180 - 2n.
   d. 180 - (r + s). e. 90. f. $90 - \frac{1}{2}k$.

2. m ∠P = 4.2.

3. The Parallel Postulate assures us that L is the only parallel to $\overleftrightarrow{AC}$ through B. It is also used to prove that alternate interior angles are equal when parallels are cut by a transversal, and this theorem in turn is used in the proof of the angle-sum theorem.

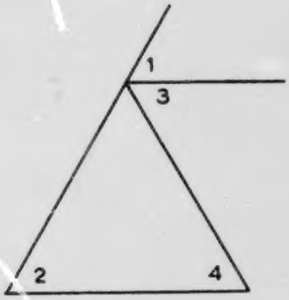
4. (Numbers in parentheses were given in the original problem.)



5. a. Yes. b. No.

6. Proof: By theorems on transversals of parallels ∠EBD ≅ ∠A and ∠DBC ≅ ∠C. But ∠EBD ≅ ∠DBC. Therefore ∠A ≅ ∠C. Hence AB = BC.

7. Proof: We have m∠1 = m∠3 by hypothesis and m∠2 = m∠4 by the base angle theorem. But m∠1 + m∠3 = m∠2 + m∠4 by Corollary 9-12-4. Taking half of each sum we have m∠1 = m∠2. Since ∠1 and ∠2 are congruent corresponding angles formed by a transversal, the bisector is parallel to the base.



8. Proof:

1. r = m∠2 + m∠4. s = m∠1 + m∠3.	1. Exterior angle theorem.
2. r + s = (m∠1 + m∠2) + (m∠3 + m∠4).	2. Addition.
3. m∠1 + m∠2 = t and m∠3 + m∠4 = u.	3. Angle addition theorem.
4. r + s = t + u.	4. Statements 2 and 3.

9. Proof:

$\triangle$s CES and CDT have $\angle C$ in common and each has a right angle as well as the congruent sides $\overline{CE}$ and $\overline{CD}$. Therefore $\triangle CES \cong \triangle CDT$ by S.A.A. and $\overline{ES}$ and $\overline{DT}$ are congruent corresponding parts.

10. Proof:

By S.A.S. the four triangles are congruent. Hence the four sides $\overline{KL}$, $\overline{LM}$, etc. are congruent. But of the three angles at N, for example, two are complementary. Therefore the third is a right angle. Likewise the other angles of KLMN are right angles and the figure is a square.

11. Proof:

Since $QB = QA$, $\angle B \cong \angle 1$.
Since $\angle 2$ and $\angle 1$ are complements, $\angle 2$ and $\angle B$ are also.
But $\angle B$ and $\angle C$ are complements, hence $\angle 2 \cong \angle C$ because complements of the same angle are congruent. Now $QA = QC$, and hence $QB = QC$.

12. In $\triangle ABC$, $m\angle B = 90 - a$.

In $\triangle ATC$, $m\angle ATS = \dfrac{180 - a}{2}$.

In $\triangle BTR$, $m\angle BTR = \dfrac{180 - (90 - a)}{2} = \dfrac{90 + a}{2}$,

$$\begin{aligned} m\angle STR &= 180 - (m\angle ATS + m\angle BTR) \\ &= 180 - \left(\frac{180 - a}{2} + \frac{90 + a}{2}\right) \\ &= 180 - 135 = 45. \end{aligned}$$

Problem Set 9-6 .

1. a. All four quadrilatersls.
   b. All four.
   c. Square, rhombus.
   d. All four.
   e. Square, rhombus.
   f. All four.
   g. Square, Rhombus.
   h. All four.
   i. Rectangle, square.
   j. Rectangle, square.

2. $m \angle A = 80$.
   $m \angle B = 100$.
   $m \angle F = 80$.
   $m \angle H = 100$.

   When two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary, and so $x + 30 + 2x - 60 = 180$. Therefore $x = 70$.

3. Proof:

   Since the opposite angles of a parallelogram are congruent, $\angle H \cong \angle A$ and also $\angle R \cong \angle A$, so that $\angle R \cong \angle H$. Since interior angles on the same side of a transversal which cuts parallel lines are supplementary, $\angle M$ is supplementary to $\angle A$. By substitution we see that $\angle M$ is supplementary to $\angle H$.

4. a. Yes. No. No. No.
   b. Yes. No. No. No.
   c. Yes. Yes. No. No.
   d. Yes. No. No. No.
   e. No. No. No. No.
   f. No. No. No. No.
   g. Yes. No. Yes. No.
   h. Yes. No. Yes. No.
   i. Yes. Yes. No. No.
   j. Yes. Yes. Yes. Yes.
   k. Yes. No. No. No.
   ℓ. Yes. No. No. No.

5. Proof:

AD = BC and AB = DC since opposite sides of a parallelogram are congruent. Then $\triangle APD \cong \triangle CRB$ and $\triangle APB \cong \triangle CRD$ by S.A.S. Then by corresponding parts $RD \cong PB$ and $PD \cong RB$. Having opposite sides congruent, DPBR is a parallelogram.

6. Proof:

1. $\overline{FE} \parallel \overline{AD}$. $\overline{FE} \parallel \overline{BC}$.	1. Opposite sides of a parallelogram are parallel.
2. $\overline{AD} \parallel \overline{BC}$.	2. If two lines in a plane are parallel to a third, they are parallel to each other.
3. $\overline{FE} \cong \overline{AD}$. $\overline{FE} \cong \overline{BC}$.	3. Opposite sides of a parallelogram are congruent.
4. $\overline{AD} \cong \overline{BC}$.	4. Step 3.
5. ABCD is a parallelogram.	5. From steps 2 and 4, it has the same two sides both parallel and congruent.

7. Proof:

1. PXRY is a ▱.	1. If opposite sides are $\parallel$, a quadrilateral is a ▱.
2. PX = RY, RX = PY.	2. Opposite sides of a ▱ have equal lengths.
3. $\angle XPS \cong \angle T$.	3. If lines are $\parallel$, corresponding angles are congruent.
4. $\angle S \cong \angle T$.	4. If two sides of a triangle are congruent, the angles opposite these sides are congruent.

7. 5. $\angle XPS \cong \angle S$. — 5. Angles congruent to the same angle. Steps 3 and 4.

6. PX = SX. — 6. If two angles of a triangle are congruent, the sides opposite these angles have equal lengths.

7. PY = TY. — 7. By steps similar to steps 2 - 6.

8. PX + XR + RY + YP = SX + XR + RY + YT, or PX + XR + RY + YP = RS + RT. — 8. By steps 6 and 7.

8. Proof:

Statements	Reasons
1. DQ = BQ.	1. Diagonals of a parallelogram bisect each other.
2. $\overline{DC} \parallel \overline{BA}$.	2. Opposite sides of a parallelogram are parallel.
3. $\angle EDQ \cong \angle FBQ$.	3. Alternate interior angles.
4. $\angle DQE \cong \angle BQF$.	4. Vertical angles.
5. $\triangle DQE \cong \triangle BQF$.	5. A.S.A.
6. EQ = FQ.	6. Corresponding parts.
7. EF is bisected by Q.	7. Definition of bisect.

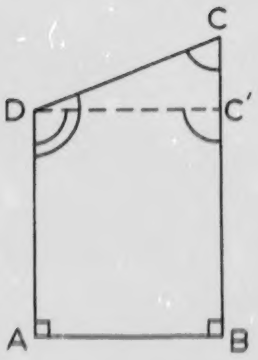
9. Through D draw a parallel to $\overline{CB}$ meeting $\overline{AB}$ at X. Then DCBX is a parallelogram in which case CB = DX. Since it was given that AD = CB, therefore DX = DA and $\angle DXA \cong \angle A$. But, by corresponding angles $\angle DXA \cong \angle B$. Therefore $\angle A \cong \angle B$.

10. Proof:

$\triangle DCQ \cong \triangle KBQ$ by A.S.A. or S.A.A. so that Q is midpoint of $\overline{DK}$. In $\triangle ADK$, $\overline{PQ} \parallel \overline{AK}$ and $PQ = \frac{1}{2} AK = \frac{1}{2}(AB + BK)$. $BK = CD$ since they are corresponding parts of congruent triangles. Hence $PQ = \frac{1}{2}(AB + CD)$.

12. Proof:

1. Draw $\overline{DB}$.	1. 2 points determine a segment.
2. $\overline{RQ} \parallel$ to and $\cong \frac{1}{2}\ \overline{DB}$.	2. A segment joining the midpoints of two sides of a triangle is parallel to the third side and has $\frac{1}{2}$ of its measure.
3. $\overline{SP} \parallel$ to and $\cong \frac{1}{2}\ \overline{DB}$.	3. Statement 2.
4. $\overline{RQ} \cong \overline{SP}$.	4. Their measures are equal to $\frac{1}{2}$ of the same quantity.
5. $\overline{RQ} \parallel \overline{SP}$.	5. 2 lines parallel to the same line are parallel to each other.
6. SPQR is a ▱.	6. One pair opposite sides parallel and congruent.
7. $\overline{SQ}$ and $\overline{PR}$ bisect each other.	7. Diagonals of ▱s bisect each other.



13. Let C′ be between B and C such that AD = BC′. Then $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC'}$ so that ABC′D is a parallelogram and $m\angle ADC' = m\angle BC'D$. Making this replacement in $m\angle ADC' < m\angle ADC$, we have $m\angle BC'D < m\angle ADC$. By the exterior angle theorem $m\angle C < m\angle BC'D$. Therefore $m\angle C < m\angle ADC$.

14. Given: $\triangle ACB$ with $AC = BC$, $\overline{PX} \perp \overline{AC}$, $\overline{PY} \perp \overline{BC}$, $\overline{BT} \perp \overline{AC}$.

To prove: $PX + PY = BT$.

Proof:

1. Draw a perpendicular, PQ, from P to $\overline{BT}$.	1. From a point outside a line, a perpendicular can be drawn to the line.
2. $\overleftrightarrow{PX} \parallel \overleftrightarrow{QT}$, and $\overleftrightarrow{XT} \parallel \overleftrightarrow{PQ}$.	2. In a plane, two lines perpendicular to the same line are parallel.
3. PQTX is a ▱.	3. By definition of parallelogram.
4. $PX = QT$.	4. Opposite sides of a parallelogram have equal lengths.
5. $\angle QPB \cong \angle A$.	5. If lines are parallel, corresponding angles are congruent.
6. $\angle YBP \cong \angle A$.	6. Base angles of an isosceles triangle are congruent.
7. $\angle QPB \cong \angle YBP$.	7. Angles congruent to the

14.

	same angle are congruent to each other.
8. $\triangle QPB \cong \triangle YBP$.	8. By the S.A.A. Theorem.
9. $PY = BQ$.	9. Corresponding sides of congruent triangles.
10. $PX + PY = QT + BQ$, or $PX + PY = BT$.	10. Steps 4 and 9.

15. Given: P interior to equilateral $\triangle ABC$. $\overline{PQ}$, $\overline{PR}$, $\overline{PS}$ and CD are perpendiculars as shown.

To prove: $PQ + PR + PS = CD$.

Proof:

1. Draw a line, through P, $\perp$ $\overline{CD}$ intersecting $\overline{AC}$, $\overline{CD}$, $\overline{BC}$ as shown.	1. From a point outside a line, one $\perp$ can be drawn.
2. $\overline{PZ} \parallel \overline{QD}$, $\overline{PQ} \parallel \overline{ZD}$.	2. Two lines $\perp$ to the same line are parallel.
3. PQDZ is a ▱.	3. By definition of parallelogram.
4. $PQ = ZD$.	4. Opposite sides of a parallelogram have equal lengths.
5. $PR + PS = CZ$.	5. By Example 14.
6. $PQ + PR + PS = CZ + ZD$, $PQ + PR + PS = CD$.	6. From steps 4 and 5.

16. EFOD is a parallelogram, by definition. Hence $EF = DO$ and $\overleftrightarrow{EF} \parallel \overleftrightarrow{DO}$. Similarly $DO = CB$ and $CB = OA$. Therefore $EF = OA$ and $\overleftrightarrow{EF} \parallel \overleftrightarrow{OA}$. So EFAO is a parallelogram and $\overleftrightarrow{FA} \parallel \overleftrightarrow{EO}$. Since $\overleftrightarrow{CD} \parallel \overleftrightarrow{EO}$, we have $\overleftrightarrow{FA} \parallel \overleftrightarrow{CD}$.

17. a. $ABB'A'$ is a parallelogram so that $AA' = BB'$. Similarly $BCC'B'$ is a parallelogram and $BB' = CC'$. Thus $AA' = CC'$ and $AA'C'C$ is a parallelogram. Hence $\overleftrightarrow{AC} \parallel \overleftrightarrow{A'C'}$.

b\. The proof does not apply if the figure is not plane because it has not been proved that if two lines in space are parallel to a third line they are parallel to each other.

18\.

1. Through C draw a line $\overleftrightarrow{CL} \parallel \overleftrightarrow{DY}$.	1. Through a point outside a line there is a parallel to the line.
2. $BC = AD$.	2. Opposite sides of a parallelogram have equal lengths.
3. $BY = DX$.	3. Halves of equal numbers are equal.
4. BYDX is a ▱.	4. If 2 sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.
5. $\overleftrightarrow{DY} \parallel \overleftrightarrow{XB}$.	5. Opposite sides of a parallelogram are parallel.
6. $\overleftrightarrow{CL} \parallel \overleftrightarrow{XB}$.	6. Two lines parallel to the same line are parallel to each other.
7. $CQ = QT$.	7. If parallels cut off equal segments on one transversal, they cut off equal segments on other transversals.
8. $AT = TQ$.	8. By steps corresponding to steps 1 - 7.

9. AT = TQ = QC. 9. By steps 7 and 8.

Problem Set 9-7.

1. Given $\overleftrightarrow{PQ}$ bisects $\overleftrightarrow{CE}$, $\overleftrightarrow{PQ} \parallel \overleftrightarrow{CD}$.

   Prove $\overleftrightarrow{PQ}$ bisects $\overline{ED}$.

   Proof:

   By theorem 9-21, $\overline{PQ'} \parallel \overline{CD}$. By the hypothesis $\overline{PQ} \parallel \overline{CD}$. This contradicts the Parallel Postulate unless $Q' = Q$, and in this case $\overline{PQ}$ bisects $\overline{ED}$.

2. a. By Theorem 9-25, RS = ST; and then by the same theorem ZY = YX.

   b. No.

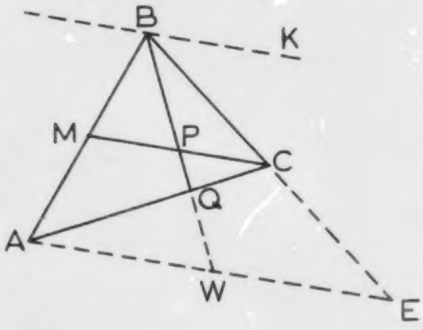
3. The right edge of sheet A is a transversal divided into congruent segments by ruled parallels. By Corollary 9-25-1, any other transversal, in particular $\overline{OQ}$, will be divided into congruent segments by the same parallels.

4. Congruent corresponding angles assure parallel lines through $N_1, N_2, \cdots, N_5$. Considering a sixth parallel through A, Corollary 9-25-1 explains why $\overline{AB}$ will be divided into equal parts.

5. Reasons:

   1. A line joining the midpoints of two sides of a triangle is parallel to the third side and half as long.
   2. If two sides of a quadrilateral are congruent and parallel, it is a parallelogram.
   3. The diagonals of a parallelogram bisect each other.

6. Extend $\overline{BC}$ making CE = BC and draw $\overline{AE}$. Extend $\overline{BP}$ to meet $\overline{AE}$ at W. Draw $\overleftrightarrow{BK} \parallel \overline{AE}$. Now $\overline{MC} \parallel \overline{AE}$ by theorem 9-21 and BP = PW by theorem 9-25. By theorem 9-21 again AW = 2MP = 2PC = WE. Hence $\overline{BW}$ is a median cf $\triangle$ ABE and meets the median $\overline{AC}$ at a point Q where AQ = 2QC.



7. a. 3.
   b. 7.
   c. 9.
   d. 500.
   e. No set of parallels can include $\overleftrightarrow{AR}$, $\overleftrightarrow{BS}$ and $\overleftrightarrow{CT}$.

Solutions of Problems

Problem Set 10-1.

1. a. 1. b. 1. c. 0. d. 1. e. 1. f. 1. g. 0.
h. 1. i. 1. j. 0. k. 1. l. 0.

2. Let $\overline{AZ}$ intersect plane n at T. Draw $\overline{AX}$, $\overline{BT}$, $\overline{TY}$, and $\overline{CZ}$. Then $\overline{BT} \parallel \overline{CZ}$ and $\overline{TY} \parallel \overline{AX}$ by Theorem 10-1. From Theorem 9-25, in plane ACZ, AT = TZ; and in plane AZX XY = YZ.

3. Proof:

Statements	Reasons
1. $s \parallel r$.	1. Given.
2. $\overline{AB} \perp r$.	2. Given.
3. $\overline{AB} \perp s$.	3. Theorem 10.2
4. $\overline{AB} \perp CX$ and $\overline{AB} \perp \overline{CY}$.	4. Theorem 8.3
5. $\Delta ACZ \cong \Delta ACY$.	5. S.A.S.
6. AX = AY	6. Corresponding parts.

4. Proof:

Statements	Reasons
1. $m \perp \overline{AB}$, $n \perp \overline{AB}$.	1. Given.
2. $m \parallel n$.	2. Theorem 10.3.
3. $m \perp \overline{CD}$.	3. Given.
4. $n \perp \overline{CD}$.	4. Theorem 10.2

5. Proof: By theorem 10-5, AB = CD. Consider $\overline{BD}$. $\overline{AB} \perp \overline{BD}$ and $\overline{CD} \perp \overline{BD}$ by definition of a line perpendicular to a plane. Then $\Delta ABD \cong \Delta CDB$ by S.A.S. and AD = CB by

corresponding parts.

6. By theorem 10-3 we know $E \parallel F$. By theorem 10-1 we know $\overleftrightarrow{AD} \parallel \overleftrightarrow{BK}$ and $\overleftrightarrow{CA} \parallel \overleftrightarrow{HB}$. Since $BK = AD$ and $BH = AC$, we know we have two parallelograms. These are rectangles since $\overline{AB}$ is perpendicular to both planes and therefore to lines in the planes through A and B. $\angle$s CAD and HBK are plane angles of the dihedral angle D-AB-H and are equal. Then $\triangle CAD \cong \triangle HBK$ by S.A.S. However, we do not know the measure of any of the angles of the two triangles and so cannot find the length of $\overline{CD}$.

7. Let points D and G be such that $AD = BG$ and E and F be such that $AE = BF$. Draw $\overline{DE}$ and $\overline{GF}$. Then:

   (1) $\overline{AE} \parallel \overline{BF}$ and $\overline{AD} \parallel \overline{BG}$ by Theorem 10-1.

   (2) AEFB and ADGB are parallelograms since they have two sides parallel and equal in length.

   (3) $EF = AB$ and $DG = AB$ because opposite sides of a parallelogram have equal lengths. Also $\overline{EF} \parallel \overline{AB}$ and $\overline{DG} \parallel \overline{AB}$.

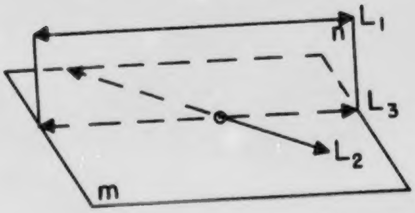
   (4) Therefore $EF = DG$ and $\overline{EF} \parallel \overline{DG}$ making EDGF a parallelogram by Theorem 9-19.

   (5) $ED = FG$.

   (6) $\triangle ADE \cong \triangle GFD$ by S.S.S.

   (7) $\angle DAE \cong \angle BGF$.

8. Given two skew lines, L_1 and L_2, at any point P on L_2 draw the one line L_3 parallel to L_1. Then L_2 and L_3 intersect and determine a plane parallel to L_1.



Proof: L_1 and L_3 are coplanar and determine a plane n because they are parallel. L_1 and L_3 cannot be coplanar because they are skew. Hence L_2 and L_3 are distinct intersecting lines determining a plane m. Planes m and n have the line L_3 in common, hence it is their intersection. L_1, which is in n, could intersect m only at some point of L_3; and this is impossible since $L_1 \parallel L_3$. Hence $L_1 \parallel m$.

9. Proof: $\overline{QP} \parallel \overline{SM}$ by Corollary 10-4-2. In the plane of $\overline{RL}$ and $\overline{QP}$, $\overline{QP} \perp \overline{PL}$; and in the plane of $\overline{SM}$ and $\overline{QP}$, $\overline{QP} \perp \overline{PM}$. Since $\overline{QP}$ is perpendicular to both PL and $\overline{PM}$, $\overline{QP} \perp n$. Then both $\overline{RL}$ and $\overline{SM}$ are perpendicular to n by Corollary 10-4-1.

Problem Set 10-2.

1. $\angle C\text{-}AB\text{-}D$, $\angle A\text{-}BC\text{-}D$, $\angle A\text{-}CD\text{-}B$, $\angle B\text{-}AD\text{-}C$, $\angle B\text{-}AC\text{-}D$, $\angle A\text{-}BD\text{-}C$.
2. Yes; Yes; Yes.
3. Yes; Yes; Yes.
4. $\angle CPB$ is a plane angle of $\angle C\text{-}PA\text{-}B$. Since $m\angle CPB = 90$, $m\angle C\text{-}PA\text{-}B = 90$. $\angle CAB = 60^\circ$ since ΔCAB can be proved to be equilateral.
5. a. 1 b. 1 c. 0 d. 0 e. 0
   f. 1 g. 0 h. 0 i. 0 j. 0 k. 1.
6. Proof: Having the lines suggested in the hint $\overline{XP} \perp r$ and $\overline{YP} \perp s$ by Corollary 10-6-2. Then $\overline{XP} \perp \overline{QP}$ and $\overline{YP} \perp \overline{QP}$ by the definition of a line perpendicular to a plane. By theorem 8-3, $\overline{QP} \perp E$.

   Since $\overline{XP} \perp m$, $\overline{XP} \perp \overline{PQ}$ and $\angle XPQ$ is a right angle. Therefore $\angle X\text{-}AB\text{-}Q$ is a right dihedral angle, and by definition of perpendicular planes $s \perp m$.
7. $x = 45$; $m = 45$; $y = 90$.
8. Case (1). Suppose the third line $\overleftrightarrow{rt}$ does have a point Q in common with either $\overleftrightarrow{rs}$ or $\overleftrightarrow{st}$. Then Q is in all three of planes r, s and t. The only points in both r and s are in $\overleftrightarrow{rs}$ and the only points in both s and t are in $\overleftrightarrow{st}$. Hence $\overleftrightarrow{rs}$ and $\overleftrightarrow{st}$ would have point Q in common, in direct

contradiction to the hypothesis of this first case. Hence the supposition that $\overleftrightarrow{rt}$ has a point in common with $\overleftrightarrow{rs}$ or $\overleftrightarrow{st}$ is false and no two of the lines $\overleftrightarrow{rs}$, $\overleftrightarrow{st}$ and $\overleftrightarrow{rt}$ intersect. Since each pair is coplanar, the three lines are parallel.

Case (2) Point P would belong to r, s and t and therefore be a point of $\overleftrightarrow{rt}$ as well as of $\overleftrightarrow{rs}$ and $\overleftrightarrow{st}$, so that the three lines would be concurrent.

9. The point $\underline{\overline{X}}$ lies in the plane (ABC) and also in the plane (A'B'C)), and hence on their intersection. Similarly, Y and Z lie on the intersection on these two planes, or the points $\underline{\overline{X}}$, Y, Z lie on a line u, which was to be proved.

Remarks: The two non-parallel planes always intersect, but it might happen that $\overleftrightarrow{B'C'}$ and $\overleftrightarrow{BC}$ are parallel lines, so there would be no point X. This would happen if and only if $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ are both parallel to the line u. This could not happen for two pairs of side lines for we could not have two lines through a vertex parallel to u.

Remark 2. The Theorem is also valid if plane (ABC) = plane (A'B'C'), but we have not proved it.

10. There is no line u as in problem 1, but $\overline{AB} \parallel \overline{A'B'}$, $\overline{BC} \parallel \overline{B'C'}$ and $\overline{AC} \parallel \overline{A'C'}$. (Theorem 10-1). The triangles ΔABC and $\Delta A'B'C'$ are similar.

Proof:

$$\frac{AB}{A'B'} = \frac{UA}{UA'} = \frac{UB}{UB'}$$ (Theorems Chapter 12) 12-1 and 12-3.

$$\frac{AC}{A'C'} = \frac{UA}{UA'}$$

$$\frac{BC}{B'C'} = \frac{UB}{UB'} .$$

Hence

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$ and by the (S.S.S.) Theorem for similarity, $\Delta ABC \sim \Delta A'B'C'$, which was to be proved.

Problem Set 10-3.

1. a. Yes. B. No. c. Yes; yes; yes.

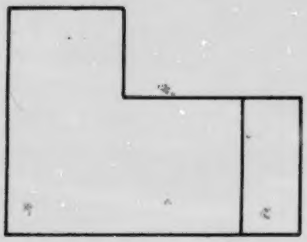
   d. Yes. e. No. f. No.

2. a. Not necessarily. b. No. c. Yes. d. Yes.

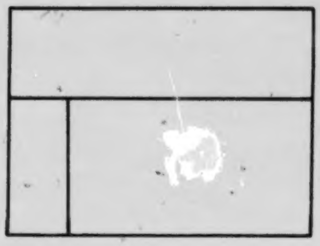
3. Proof: $\overline{AX}$ and $\overline{BY}$ are perpendicular to plane m. Hence $\overline{AX} \parallel \overline{BY}$ and ABYX is a plane figure. Since the projection of a segment is a segment (or a point) N is in this plane. Since $\overline{MN} \perp m$, $\overline{MN} \parallel \overline{AX}$ and $\overline{MN} \parallel \overline{BY}$. Then XN = NY so that N is the midpoint of $\overline{AB}$ because parallels which intercept congruent segments on one transversed intercept congruent segments on any transversal.

4.

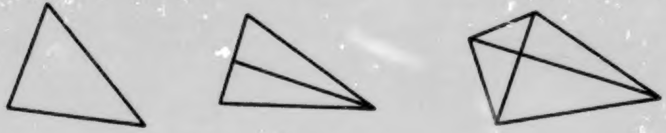
a.



b.



5. Since the intersection point shown within the large triangle in the problem may be on a vertex, on an edge or the extension of an edge, or elsewhere in the exterior of the large triangle the projection may appear as follows:



6. Proof: Let $\overline{BE}$ be the perpendicular to plane m at B. Then $\overline{BE} \perp \overline{AB}$ or $\overline{AB} \perp \overline{BE}$, and it is given that $\overline{AB} \perp \overline{BC}$. Hence $\overline{AB} \perp$ plane EBC. By definition of projection $\overline{CD} \perp m$. Then $\overline{CD} \parallel \overline{BE}$ so that D is in the plane EBC. Then $\overline{DB}$ is in this plane and $\overline{AB} \perp \overline{BD}$ or $\angle ABD$ is a right angle.

7. Proof: By definition of projection $\overline{QQ'} \perp m$ and therefore also $\overline{QQ'} \perp \overline{Q'X}$ so that $\Delta QQ'X$ is a right triangle. Then $QQ' < QX$. But $\overline{AQ} \cong \overline{AQ}$, and $\overline{AX} \cong \overline{AQ'}$ by construction. In triangles QAX and QAQ', $m\angle QAQ' < m\angle QAX$ by theorem 7-6.

8. The projection is a regular hexagon with segments from its vertices to its center.

Solutions of Problems

Problem Set 11-1.

1. The smallest number of triangular regions in each case is: In a, c, d, e, f the number of triangular region is (n-2), where n is the number of sides. In b the number of triangular regions is the same as the number of sides.

   a. 8.
   b. 8.
   c. 2.
   d. 11.
   e. 5.
   f. 2.

2. 1800 tiles.

3. 792 square inches.

4. a. False. A triangle is not a region at all, but is a figure consisting of segments.
   b. False. See Postulate 17.
   c. True. By Postulate 17.
   d. True. Postulate 18.
   e. False. If the regions overlap, their union is less than their sum.
   f. True. Since a square is a rectangle.
   g. False. The region is the union of a trapezoid and its interior.
   h. True. A triangular region is the union of one or more triangular regions.

5. a. 4.
   b. 1/4.
   c. 0.1.

d. 0.002.

e. and f. Since $\sqrt{2}$ and $\sqrt{3}$ are irrational, the base and altitude in each case are incommensurable. Hence the rectangular regions cannot be divided exactly into squares.

6. a. $f - e + v = 7 - 12 + 7 = 2$.

b. $f - e + v = 7 - 17 + 12 = 2$.

c. The computation always results in 2.

d. The computation is not affected, since the additional four edges, three faces, and one vertex results in zero being added to the total.

e. No change.

Problem Set 11-2.

1. a. Area $\triangle ABC = \frac{1}{2} \cdot 7 \cdot 24 = 84$.

b. $84 = \frac{1}{2}\ 25h$. $h = 6\frac{18}{25}$

2. 14.4 and 24.

3. a. $BC = 12$. c. $AB = 15$.

b. $CD = 6\frac{9}{11}$. d. $AE = \frac{ch}{a}$.

4. Area $\triangle CQB$ = Area $\triangle DQB$, since $CQ = DQ$ and the triangles have the same altitude, the perpendicular segment from B to $\overline{CD}$. Area $\triangle AQC$ = Area $\triangle DQA$, since $CQ = DQ$ and the triangles have the same altitude, the perpendicular segment from A to $\overline{CD}$. Adding, we have Area $\triangle ABC$ = Area $\triangle ABD$.

Alternate Proof: Draw $\overline{CE} \perp \overline{AB}$ and $\overline{DF} \perp \overline{AB}$. Then $\triangle CEQ \cong \triangle DFQ$ by A.A.S., and $CE = DF$. Since $\triangle ABC$ and $\triangle ABD$ have the same base and their altitudes have equal lengths, the triangles have equal areas.

5. The area of the square is s^2.

The area of each of the four triangles is $\frac{1}{2}$ bs.

Hence, the area of the star is $s^2 - 2$ bs.

6. a. 6. c. $18\frac{2}{3}$.

b. 12. d. Since GB and AF are measures of the same altitude, there is not enough information given to determine an unique answer.

7. Since a diagonal of a parallelogram divides it into two congruent triangles, Area $\triangle$AFH is equal to half the area of the parallelogram. Area $\triangle$AQH = Area $\triangle$FQH since the bases, $\overline{AQ}$ and $\overline{QF}$, are congruent and the triangles have the same altitude, a perpendicular from H to $\overline{AF}$. Each is then one-fourth of the area of the parallelogram. In the same way it can be shown that Area $\triangle$ABQ = Area $\triangle$FBQ.

8. a. 36. d. 154.

b. 21. e. $121\frac{1}{2}$.

c. 55.

9. 98.

10. $\frac{A}{A'} = \frac{\frac{1}{2}bh}{\frac{1}{2}bh'} = \frac{h}{h'}$.

11. Since the three triangles have the same base, by the theorem of Problem 10 their areas are in proportion to their altitudes.

12. Area of triangle = $\frac{1}{2}$ bh.

Area of parallelogram = bh'.

$\frac{1}{2}bh = bh'$.

$h = 2h'$.

The altitude of the triangle is twice the altitude of the parallelogram.

13. a. Area ▱ABCD is twice Area △BCE because the figures have the same base ($\overline{BC}$) and equal altitudes, since $\overline{AE} \parallel \overline{BC}$.

b. The two areas are equal.

c. The areas are equal because the bases ($\overline{AF}$ and $\overline{FD}$) are congruent and their altitudes are congruent since $\overline{AD} \parallel \overline{BC}$.

d. Area △CFD = $\frac{1}{2}$ (Area △BCE) since FD = $\frac{1}{2}$ BC and the two triangles have equal altitudes. Therefore, Area ▱ABCD = 2 (Area △BCE) = 4 (Area △CFD).

14. The area of trapezoid DFEC = 34.

The area of trapezoid AGFD = 165.

And so, area of AGECD = 199.

Area △AGB = 30.

Area △BCE = 32 $\frac{1}{2}$.

Subtracting the sum of the areas of the two triangles from the area of AGECD, we have 136 $\frac{1}{2}$. The area of the field is 136$\frac{1}{2}$ sq. rods.

15. Given: Figure ABCD with $\overline{AC} \perp \overline{DB}$.

Prove: Area of ABCD = $\frac{1}{2}$ AC · DB.

Proof: Area of ABCD = Area △ACD + Area △ABC by Postulate 19.

But Area △ACD = $\frac{1}{2}$ AC · DP and Area △ABC = $\frac{1}{2}$ AC·PB.

Therefore, Area of ABCD = $\frac{1}{2}$ AC·DP + $\frac{1}{2}$ AC·PB

= $\frac{1}{2}$ AC (DP + PB) = $\frac{1}{2}$ AC·DB.

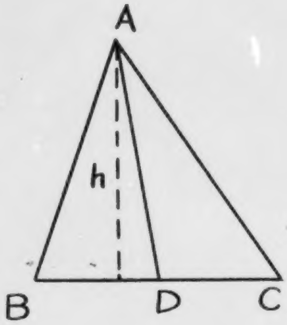
16. The area of a rhombus equals one-half the product of the lengths of its diagonals.

17. 12.

18. $A = \frac{1}{2} dd' = 150 = bh = 12b$.
therefore $b = 12\frac{1}{2}$. The area is 150; the length of a side is $12\frac{1}{2}$.

19. Yes. The proof would be the same as for Problem 15 with each "+" replaced by "-".

20. All three triangles have the same altitude. Hence, since BD = DC, the two smaller triangles have equal area, by Theorem 11-6, and each is one-half the area of the big Triangle, by Theorem 11-5.



21. a. By the previous problem,
Area △ABE = Area △BAD = $\frac{1}{2}$ (Area △ABC). Subtracting Area △ABG from each leaves Area △AEG = Area △BDG.
Since the medians are concurrent, the third median, with the other two, divides the triangle into six triangles:
Area △AEG = Area △BDG,
Area △CGE = Area △BGF, and
Area △CGD = Area △AGF. But Area △BDG = Area △CGD by Theorem 11-6, and consequently all the areas are equal.
Therefore, Area △BDG = $\frac{1}{6}$ (Area △ABC).

22. Since AB is constant, the altitude, to $\overleftrightarrow{AB}$ must be constant, by Theorem 11-6.

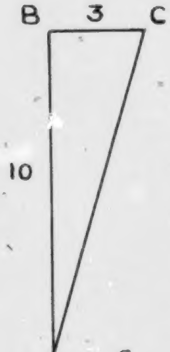
Call the length of the altitude, from P to $\overleftrightarrow{AB}$, h. Then in plane E, P may be any point on either of the two lines parallel to $\overleftrightarrow{AB}$ at a distance h from $\overleftrightarrow{AB}$. In space, P may be any point on a cylindrical surface having $\overleftrightarrow{AB}$ as its axis and h as its radius.

23. a. 104.

b. $\frac{1}{2}\cdot 16\cdot 13 = 104$.

c. Are you puzzled too?

Problem Set 11-3a.

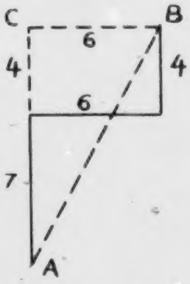


1. $(AC)^2 = 100 + 9$.

$= 109$.

$AC = \sqrt{109}$.

He is $\sqrt{109}$ miles from his starting point. (Between 10.4 and 10.5 miles.)

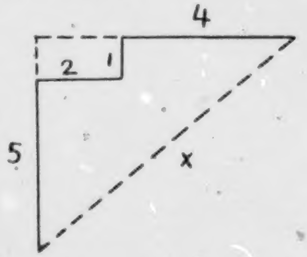


2. The single right triangle ACB serves our purpose here.

$(AB)^2 = (11)^2 + (6)^2 = 157$.

$AB = \sqrt{157}$.

He is approximately 12.5 miles from his starting point.



3. $(6)^2 + (6)^2 = x^2$.

$72 = x^2$.

$6\sqrt{2} = x$.

He is approximately 8.5 miles from his starting point.

4. In right $\triangle ABE$,

$(AE)^2 = (AB)^2 + (BE)^2 = 4^2 + 3^2 = 25$,

$AE = 5$.

In right $\triangle AED$,

$(AD)^2 = (AE)^2 + (ED)^2 = 5^2 + 12^2 = 169$,

$AD = 13$.

5. a, c, d, e.

6. a. It is sufficient to show that $(m^2-n^2)^2 + (2mn)^2 = (m^2+n^2)^2$.

$(m^2-n^2)^2 + (2mn)^2 = m^4 - 2m^2n^2 + n^4 + 4m^2n^2 =$

$m^4 + 2m^2n^2 + n^4 = (m^2 + n^2)^2$.

b. $m = 2$, $n = 1$ gives sides with lengths (3, 4, 5.).

Similar triangles have sides with lengths

(6, 8, 10), (9, 12, 15), (12, 16, 20) and (15, 20, 25).

$m = 3$, $n = 2$ gives (5, 12, 13).

$m = 4$, $n = 1$ gives (15, 8, 17)

$m = 4$, $n = 3$ gives (9, 24, 25).

7. a. $AY = \sqrt{2}$, $AZ = \sqrt{3}$. $AB = \sqrt{4} = 2$.

b. $AC = \sqrt{5}$. Next segment has length $= \sqrt{6}$.

8. a. $AC = \sqrt{8}$ or $2\sqrt{2}$.

b. $(AY)^2 = (AC)^2 + (YC)^2$, from which $AY = 3$.

9. a. $h_c^2 = 13^2 - x^2 = 169 - x^2$;

also $h_c^2 = 15^2 - (14 - x)^2 = 225 - 196 + 28x - x^2$.

Eliminating h_c^2:

$169 - x^2 = 29 + 28x - x^2$.

$28x = 140$.

$x = 5$,

$h_c = 12.$

b. $h_a = 14^2 - x^2 = 196 - x^2;$

also $h_a^2 = 13^2 - (15 - x)^2 = 169 - 225 + 30x - x^2.$

Eliminating h_a^2:

$196 - x^2 = -56 + 30x - x^2.$

$30x = 252.$

$x = 8.4.$

$h_a = 11.2$

10. Let $\overleftrightarrow{CD}$ meet $\overleftrightarrow{AB}$ at D. Let $BD = x.$

$h_c^2 = 14^2 - x^2 = 196 - x^2,$

also $h_c^2 = 18^2 - (6 + x)^2 = 324 - 36 - 12x - x^2.$

Eliminating h_c^2:

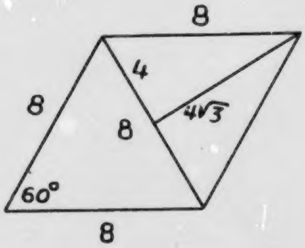
$196 - x^2 = 288 - 12x - x^2.$

$12x = 92.$

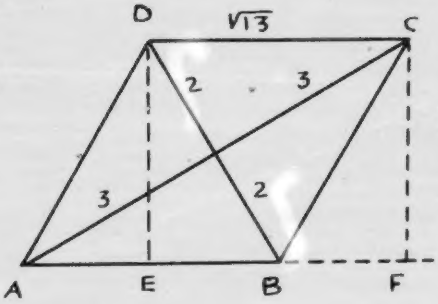
$x = 7\frac{2}{3}.$

$h_c = \frac{1}{3}\sqrt{1235}.$ (approx. 11.71.)

11. The shorter diagonal divides the rhombus into two equilateral triangles. Hence its length is 8. Since the diagonals are perpendicular bisectors of each other we can use the Pythagorean Theorem to get the length of the longer diagonal equal to $8\sqrt{3}$.



12. Since the sides are all congruent, and the area of the rhombus is the product of the measures of any side and its corresponding altitude, then all the altitudes are congruent. Hence, it is sufficient to find one altitude. The diagonals bisect each other at right angles. Hence, each side has length $\sqrt{13}$. Then,



Area of $\triangle$ ABD $= \frac{1}{2} \cdot 4 \cdot 3 = 6 = \frac{1}{2}$ DE $\sqrt{13}$,

and DE $= \frac{12}{13}\sqrt{13}$.

13. By the Pythagorean Theorem, AB = 13.
The area of $\triangle$ ABC $= \frac{1}{2} \cdot 13h = \frac{1}{2} \cdot 5 \cdot 12$.
Hence $13\,h = 5 \cdot 12$ and $h = \frac{60}{13} = 4\frac{8}{13}$.

14. By the Pythagorean Theorem, AB = 17.
The area of $\triangle$ ABC $= \frac{1}{2}\,17h = \frac{1}{2}\,15 \cdot 8$.
Hence $17h = 15 \cdot 8$ and $h = \frac{120}{17} = 7\frac{1}{17}$.

15. Area $\triangle$ ABC $= \frac{1}{2}$ ch, and

$h = \frac{2(\text{Area } \triangle \text{ ABC})}{c}$. But Area $\triangle$ ABC $= \frac{1}{2}\,ab$,

and $c = \sqrt{a^2 \text{ x } b^2}$. Therefore, $h = \frac{ab}{\sqrt{a^2 + b^2}}$.

16. (Individual answers).

17. Since $\triangle$ ABC $\cong \triangle$ BED, m$\angle$BAC = m$\angle$EBD. But $\angle$BAC is complementary to $\angle$ABC, so $\angle$EBD is complementary to $\angle$ABC. Since $\angle$EBD + $\angle$EBA + $\angle$ABC = 180, then $\angle$EBA = 90. Now,

Area of CAED = Area $\triangle ABC$ + Area $\triangle AEB$ + Area $\triangle BED$.

$$\frac{1}{2}(a+b)(a+b) = \frac{1}{2}ab + \frac{1}{2}c^2 + \frac{1}{2}ab.$$

$$a^2 + 2ab + b^2 = 2ab + c^2.$$

$$a^2 + b^2 = c^2.$$

18. Draw $\overline{RD}$.

From $RB^2 + RD^2 = BD^2$, $RD = \sqrt{3}$.

From $RS^2 + SD^2 = RD^2$, $RS = \sqrt{2}$.

19. By Pythagorean Theorem, $AC = \sqrt{2}$. Therefore $CD = \sqrt{2}$ and $BD = 1 + \sqrt{2}$. Hence, $(AD)^2 = 1 + (1 + \sqrt{2})^2$

$= 4 + 2\sqrt{2}$.

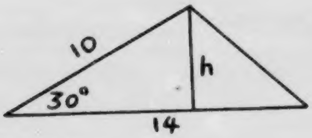
Then $AD = \sqrt{4 + 2\sqrt{2}}$.

Since $AC = CD$, $m\angle ADC = M\angle CAD$. But $m\angle ADC + m\angle CAD = x = 45$.

Then $2(m\angle ADC) = 45$, and $m\angle ADC = 22\frac{1}{2}$. $m\angle DAB = 67\frac{1}{2}$.

Problem Set 11-3b.

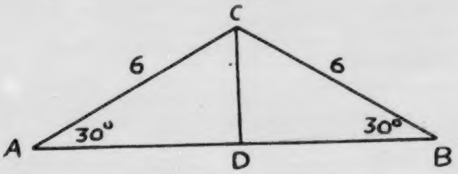
1. 5. 35.



2. Draw $\overline{CD} \perp \overline{AB}$.

Then $AD = DB = 3\sqrt{3}$.

$AB = 6\sqrt{3}$.



3. Let x = the length of the shorter leg. Since the triangle is a 30° - 60° triangle,

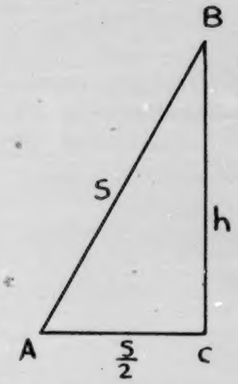
$$(2x)^2 - x^2 = 75.$$

$$3x^2 = 75.$$

$$x^2 = 25.$$

$$x = 5.$$

The length of the hypotenuse is 10.



4. By Theorem 11-10, $AC = \frac{s}{2}$

Since $(AC)^2 = (BC)^2 = (AB)^2$, we have,

$\left(\frac{s}{2}\right)^2 + h^2 = s^2$ from which $h^2 = s^2 - \frac{s^2}{4}$.

$h^2 = \frac{3s^2}{4}$ so $H = \frac{s}{2}\sqrt{3}$.

5. Since $m\angle B = 60$, then $m\angle D = 60$, and $DF = \frac{3}{2}$. Then $AF = \frac{3}{2}\sqrt{3}$.

6. $\frac{s}{2}\sqrt{3} = 15.$

$s = \frac{30}{\sqrt{3}}$

$s = 10\sqrt{3}$. A side is $10\sqrt{3}$ inches long.

7. $\frac{1}{2}$; 2; $\frac{\sqrt{3}}{3}$; $\sqrt{3}$; $\frac{\sqrt{3}}{2}$ $\frac{2\sqrt{3}}{3}$; Yes.

8. a. $\frac{1}{2}$ base $= 10\sqrt{3}$, altitude $= 10$.

$10 \cdot 10\sqrt{3} = 100\sqrt{3}.$

Area is $100\sqrt{3}$ sq. in.

b. $\frac{1}{2}$ base $= 10\sqrt{2}$, altitude $= 10\sqrt{2}$.

$10\sqrt{2} \cdot 10\sqrt{2} = 200.$

Area is 200 sq. in.

c. $\frac{1}{2}$ base $= 10$, altitude $= 10\sqrt{3}$.

$10 \cdot 10\sqrt{3} = 100\sqrt{3}.$

Area is $100\sqrt{3}$ sq. in.

9. a. $\frac{1}{2}$ base $= 12$, $h = 12$. Area is 144 sq. in.

b. $\frac{1}{2}$ base $= 12$, $h = 4\sqrt{3}$. Area is $48\sqrt{3}$ sq. in.

c. $\frac{1}{2}$ base $= 12$, $h = 12\sqrt{3}$. Area is $144\sqrt{3}$ sq. in.

10. a. $a = 30.$
$2a = 60.$
$3a = 90.$
$x = 6.$
$y = 3\sqrt{3}.$

b. $a = 30.$
$x = 5\sqrt{3}.$
$y = 10.$

c. $a = 45.$
$2a = 90.$
$x = 5.$
$y = 5\sqrt{2}.$

d. $a = 45.$
$x = 4.$
$y = 4\sqrt{2}.$

e. $x = 2\sqrt{3}.$
$y = 4.$

f. $x = 4\sqrt{2}.$
$y = 4\sqrt{2}.$

g. $a = 60.$
$x = 3\sqrt{3}.$

h. $a = 45.$
$x = 2\sqrt{2}.$

11. $FB = 3$; $HF = 3\sqrt{3}$; $AH = 6\sqrt{2}$; $AF = 3\sqrt{3}$; $m\angle ABF = 90$; $m\angle ABH = 90$; $m\angle HFB = 90$; $m\angle HBF = 60$; $m\angle BHA = m\angle BAH = 45.$ Also, $m\angle FAH = m\angle FHA.$

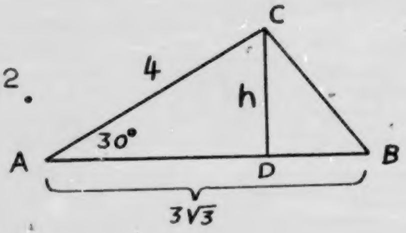
12. Let $\overline{CD}$ be the altitude to $\overline{AB}$. Let $AD = x$, $CH = h$, $BC = a$, $DB = y$. In 30°- 60° right $\triangle ACD$, $h = \frac{1}{2}\cdot 4 = 2$, $x = 2\sqrt{3}.$

Therefore $y = 3\sqrt{3} - 2\sqrt{3} = \sqrt{3}$

In right $\triangle DBC$, $a^2 = h^2 + y^2 = 4 + (\sqrt{3})^2 = 7.$

$$a = \sqrt{7}.$$

No, since $(4)^2 + (\sqrt{7})^2 \neq (3\sqrt{3})^2.$



13. Let $\overline{CD}$ be the perpendicular from C to $\overleftrightarrow{AB}$. Let $CD = h$, $BD = r$, $BC = a$.

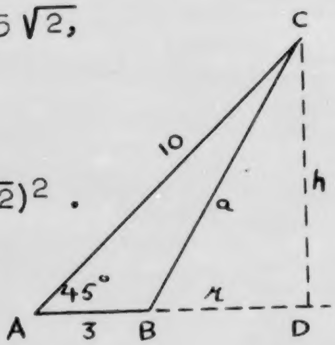
In 45°-45°-90° $\triangle ACD$, $h = AD = \frac{1}{2}\sqrt{2}\,.10 = 5\sqrt{2},$

$r = AD - 3 = 5\sqrt{2} - 3.$

In right $\triangle BDC$,

$$a^2 = r^2 + h^2 = (5\sqrt{2} - 3)^2 + (5\sqrt{2})^2 .$$
$$= 50 - 30\sqrt{2} + 9 + 50$$
$$= 109 - 30\sqrt{2}$$
$$a = \sqrt{109 - 30\sqrt{2}} .$$

BC is approximately 8.2.



14. By Pythagorean Theorem, the altitude equals 24.

The area is 240 sq. in.

15.

Statements	Reasons
1. $\triangle$s DFB and CFA are right triangles.	1. Given.
2. FD = FC. DE = CA.	2. Given.
3. $\triangle$ FDB $\cong$ $\triangle$ FCA.	3. Theorem 11-9.
4. FB = FA.	4. Corresponding parts.
5. $\triangle$ FAB is isosceles.	5. Definition of isosceles triangle.

16.

Statements	Reasons
1. AE = BF.	1. Given.
2. EF = FE.	2. Identity.
3. AF = BE.	3. Addition of steps 1 and 2.
4. DF = CE.	4. Given.
5. $\triangle$ AFD and $\triangle$ BEC are right triangles	5. Given.
6. $\triangle$ AFD $\cong$ $\triangle$ BEC.	6. Theorem 11-9.
7. $\angle$ AFD $\cong$ $\angle$ BEC.	7. Corresponding parts.
8. $\angle x \cong \angle y$.	8. Supplements of congruent angles are congruent.

17. Area $\triangle$ ABC = $\frac{1}{2}$ sh.

But by the Pythagorean Theorem, $h = \frac{s}{2}\sqrt{3}$.

Substituting, Area $\triangle$ ABC = $\frac{s}{2}\left(\frac{s}{2}\sqrt{3}\right) = \frac{s}{4}^{2}\ \sqrt{3}$.

18. a. $\sqrt{3}$. c. $\frac{3}{4}\sqrt{3}$.

b. $16\sqrt{3}$. d. $\frac{49}{4}\sqrt{3}$.

19. Let s be the length of a side.

$$\frac{s^2}{4}\sqrt{3} = 9\sqrt{3}.$$

$$s^2 = 4 \cdot 9.$$

$$s = 2 \cdot 3 = 6.$$

$$h = \frac{s}{2}\sqrt{3} = 3\sqrt{3}.$$

20. Let s be the length of a side.

$$\frac{s^2}{4}\sqrt{3} = 16\sqrt{3}.$$

$$s^2 = 4 \cdot 16.$$

$$s = 2 \cdot 4 = 8.$$

$$h = \frac{s}{2}\sqrt{3} = 4\sqrt{3}.$$

21. A side of the square is 9, and so its perimeter is 36. Then a side of the equilateral triangle is 12. The area of the equilateral triangle equals $36\sqrt{3}$.

22. $AC = 9\sqrt{2}$.

$AF = 9\sqrt{2}$.

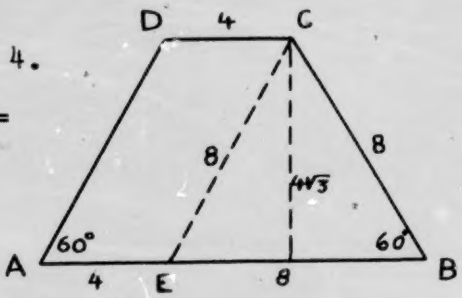
$FC = 9\sqrt{2}$.

Therefore $\triangle FAC$ is equilateral and $m\angle FAC = 60$.

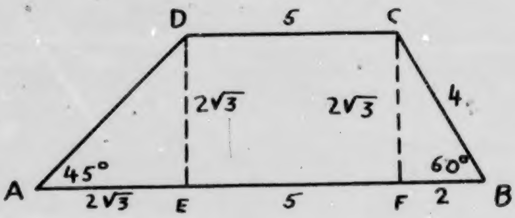
Area $\triangle FAC = \frac{(9\sqrt{2})^2}{4}\sqrt{3} = \frac{81}{2}\sqrt{3}$.

23. Make $\overline{CE} \parallel \overline{DA}$, making equilateral $\triangle EBC$ with side of 8. The altitude is $4\sqrt{3}$. Since AB = 12, AE = 4 and DC = 4. Hence, area of trapezoid ABCD =

$$\frac{1}{2}\left(4\sqrt{3}\right)(16) = [illegible]2\sqrt{2}.$$



24. Draw altitudes $\overline{DE}$ and $\overline{CF}$. Since CB = 4, FB = 2 and CF = $2\sqrt{3}$, then DE = $2\sqrt{3}$ and AE = $2\sqrt{3}$, so AB = 7 + $2\sqrt{3}$.



Therefore,

Area of ABCD = $\frac{1}{2}\left(2\sqrt{3}\right)\left(12 + 2\sqrt{3}\right) = 6 + 12\sqrt{3}$.

25. By Pythagorean Theorem AH = 15 and EC = 12.

Area $\triangle$ AHG = 150.

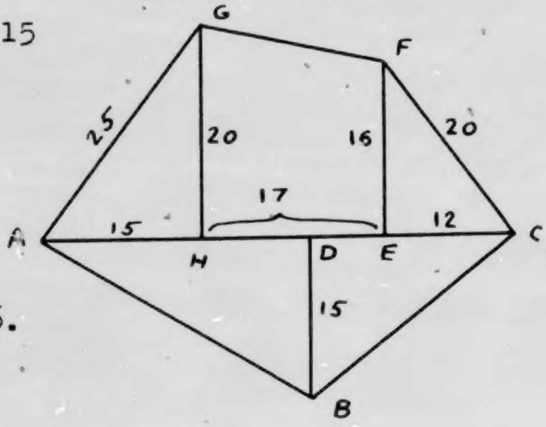
Area $\triangle$ ABC = 330.

Area $\triangle$ FEC = 96.

Area of trapezoid GHEF = 306.

Total area = 882.

Area = 882 sq. rds.



26. Since $\overline{CG} \perp$ plane E, then $\overline{CG} \perp \overline{AG}$ and $\overline{CG} \perp \overline{DG}$. $m\angle CAG = 45$, so $\triangle CAG$ is an isosceles right triangle, and CG = AG = 6. Also, AC = $6\sqrt{2}$. In $\triangle ACD$, AC = $6\sqrt{2}$. AD = $2\sqrt{6}$, so by Pythagorean Theorem, DC = $4\sqrt{3}$. In $\triangle AGD$, AG = 6, AD = $2\sqrt{6}$, so DG = $2\sqrt{3}$. Therefore DG = $\frac{1}{2}$ DC, so $m\angle DCG = 30$, and $m\angle CDG = 60$. Hence, $m\angle F\text{-}AB\text{-}E = 60$.

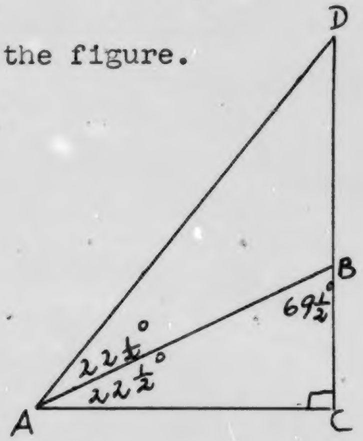
27. a. In ΔDAE and ΔDAF, $m\angle DAE = m\angle DAF$, $m\angle AED = m\angle AFD$, and $DA = DA$. Therefore, $\Delta DAE \cong \Delta DAF$ by A.A.S., and $DE = DF$.

b. $\frac{\text{Area } \Delta DAB}{\text{Area } \Delta DAC} = \frac{AB}{AC}$, since the altitudes are equal.

Also, $\frac{\text{Area } \Delta DAB}{\text{Area } \Delta DAC} = \frac{BD}{DC}$, since the triangles have the same altitude.

Therefore, $\frac{BD}{DC} \quad \frac{AB}{AC}$.

28. Let ΔACD be a 45°-45°-90° triangle as in the figure.



Then $CD = 1$, $AD = \sqrt{2}$.

By Problem 27, $\frac{BC}{BD} = \frac{1}{\sqrt{2}}$,

$$\frac{a}{1-a} = \frac{1}{\sqrt{2}},$$

$$\sqrt{2}\,a = 1 = a,$$

$$(\sqrt{2} + 1)a = 1,$$

$$a = \frac{1}{\sqrt{2} + 1} = \sqrt{2} - 1.$$

By the Pythagorean Theorem,

$$c^2 = 1 + (\sqrt{2} - 1)^2 = 1 + (2 - 2\sqrt{2} + 1),$$

$$= 4 - 2\sqrt{2},$$

$$c = \sqrt{4 - 2\sqrt{2}}.$$

29. a. In right ΔADM, $DM = \frac{e}{2}$, so $AM = \frac{\sqrt{3}}{2}e$. In right ΔAMN, $AN = \frac{e}{2}$.

By the Theorem of Pythagorus,

$$(NM)^2 = \left(\frac{e}{2}\sqrt{3}\right)^2 - \left(\frac{e}{2}\right)^2 \qquad \text{Hence, } NM = \frac{\sqrt{2}}{2}e.$$

b. $\triangle AHC \cong \triangle AHD$ by Hypotenuse-Leg, and therefore HC - HD. Then H must lie on the perpendicular bisector of $\overline{CD}$. Since in an equilateral triangle the perpendicular bisector the median, and the altitude to any side are the same, H lies on median $\overline{BM}$. Similarly, H must lie on the medians from D and C.

Hence $BH = \frac{2}{3} BM$. But $BM = AM = \frac{\sqrt{3}\,e}{2}$, so $BH = \frac{\sqrt{3}}{3} e$. Finally, in $\triangle ABH$,

$$(AH)^2 = (AB)^2 - (BH)^2 = e^2 - \left(\frac{\sqrt{3}}{3} e\right)^2 = \frac{2}{3} e^2$$

Hence, $AH = \frac{\sqrt{6}\,e}{3}$.

30. $\overline{YA} \perp \overline{AB}$ and $\overline{DA} \perp \overline{AB}$ because of the given square and rectangle. By definition $\angle YAD$ is the plane angle of $\angle X\text{-}AB\text{-}E$ and hence $m\angle YAD = 60$. By definition of projection $YD \perp E$ and hence $m\angle ADY = 90$ Then $m\angle AYD = 30$ and $AD = \frac{1}{2} AY$. Therefore area $ABCD = \frac{1}{2}$ area $ABXY = 18$.

Problem Set 12-1.

1. a. $1, \frac{7}{3}, 4.$
   b. $1, \frac{7}{3}, 4.$
   c. $1, \frac{7}{3}, 4.$

The three new sequences are identical, so the original three sequences are proportional.

2. a and d; $\frac{1}{2}$.
   a and i; 2.
   d and i; 4.
   b and f; $\frac{1}{3}$.
   b and h; 15.
   f and h: 45.
   c and e; 2.
   c and g; 3.
   e and g; $\frac{3}{2}$

   a. $1, \frac{7}{5}, \frac{9}{5}.$
   b. 1, 2, 3.
   c. $1, \frac{7}{9}, \frac{17}{9}.$
   d. $1, \frac{7}{5}, \frac{9}{5}.$
   e. $1, \frac{7}{9}, \frac{17}{9}.$
   f. 1, 2, 3.
   g. $1, \frac{7}{9}, \frac{17}{9}.$
   h. 1, 2, 3.
   i. $1, \frac{7}{5}, \frac{9}{5}.$

3. 10.

4. $\frac{1}{10}$.

5. $x = 12$, $y = 16$, $z = 44$.

6. $w = 2$, $v = \frac{5}{2}$.

7. $p = 18$, $q = 24$, $t = 70$.

8. J, k, m ~ J', k', m'.

9. b and f are correct.

10. a. Yes. b. Yes.

11. $RS = \frac{ST \cdot PQ}{QN}$.

12. AB, BC ~ A'B', B'C', or
    AB, A'B' ~ BC, B'C', etc.

Problem Set 12-2.

1. TSR ~ ABC; TS, TR, SR ~ AB, AC, BC; $\frac{2}{3}$.

   TSR ~ ACB: TS, TR, SR ~ AC, AB, CB; $\frac{2}{3}$.

   JHI ~ OQP: JH, JU, HI ~ OQ, OP, QP; 2.

   DEFG ~ KLMN; DE, EF, FG, GD ~ KL, LM, MN, NK; 3.

   DEFG ~ LKMN; DE, EF, FG, GD ~ LK, KN, NM, ML; 3.

   DEFG ~ NMLK; DE, EF, FG, GD ~ NM, ML, LK, KN; 3.

   DEFG ~ MNKL; DE, EF, FG, GD ~ MN, NK, KL, LM; 3.

2. a, b; 3, 4, 6 ~ 6, 8, 12; 2.

   a, d; 3, 4, 6 ~ 9, 12, 18; 3.

   b, d; 8, 6, 12 ~ 12, 9, 18; $\frac{3}{2}$.

   c, e; 8, 4, 9 ~ 4, 2, $4\frac{1}{2}$; $\frac{1}{2}$.

3. a. 1, 3, 4, 5, 6, 7, 8, 11; 9, 10.

   b. 2, 12.

   c. Mary had the four possible similarities in her list.

4. 8. A can correspond to any of the eight vertices of the larger figure. The eight similarities are:

   ABCDEFGH ~ MNOPRSTU.

   ABCDEFGH ~ NMPOSRUT.

   ABCDEFGH ~ OPMNTURS.

   ABCDEFGH ~ PONMUTSR.

   ABCDEFGH ~ SRUTNMPO.

   ABCDEFGH ~ RSTUMNOP.

   ABCDEFGH ~ UTSRPONM.

   ABCDEFGH ~ TURSOPMN.

5. There are two necessary relationships:

   (1) all corresponding angles are congruent, and

(2) all corresponding sides are proportional.

(Note to teacher: Fewer relationships may be sufficient to establish similarity, but these are stated later as theorems, proved on the basis of the definition.)

6. 2, 1.6 $\sim$ 7.5, h.

$h = \frac{7.5 \cdot 1.6}{2}$.

$h = 60$.

The height of the object in the enlargement is 6 inches.

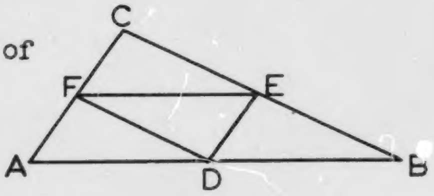
7. Yes, If $\triangle ABC \cong \triangle A'B'C'$, the conditions necessary for a similarity are met. That is,

(1) $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$ and

(2) $\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC} = 1$.

8. Given: $\triangle ABC$; D, E, F the midpoints of the sides $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ respectively.

Prove: $\triangle EFD \sim \triangle ABC$.



Proof: By Theorem 9-21, $ED = \frac{1}{2}AC$, $FE = \frac{1}{2}AB$, $FD = \frac{1}{2}CB$, and $\overline{ED} \parallel \overline{AC}$, $\overline{FE} \parallel \overline{AB}$, $\overline{FD} \parallel \overline{CB}$. FDEC, ADEF, DBEF are parallelograms. By Theorem 9-14, $\angle FDE \cong \angle BCA$, $\angle DEF \cong \angle CAB$, $\angle EFD \cong \angle ABC$; since we have also proved above that ED, FE, FD $\sim$ AC, AB, CB, $\triangle EFD \sim \triangle ABC$ by definition of similarity.

Problem Set 12-3.

1. a. $x = 1\frac{1}{2}$. b. $x = 8\frac{3}{4}$. c. $x = 16\frac{1}{4}$. d. $x = 16\frac{1}{2}$.

2. a. $a = \frac{6bc}{5d}$. b. $a = \frac{22bd}{35c}$.

c. $a = \frac{21bd}{20c}$.   d. $a = \frac{12cd}{5b}$.

3. b. $\frac{c}{a} = \frac{d}{b}$.

$ad = bc$.

$\frac{ad}{ac} = \frac{bc}{ac}$.

$\frac{d}{c} = \frac{b}{a}$.

c. $\frac{c}{a} = \frac{d}{b}$.

$bc = ad$.

$\frac{bc}{ac} = \frac{ad}{ac}$.

$\frac{b}{a} = \frac{d}{c}$

d. $\frac{c}{a} = \frac{d}{b}$.

$\frac{c}{a} + 1 = \frac{d}{b} + 1$.

$\frac{c+a}{a} = \frac{d+b}{b}$.

e. $\frac{c}{a} = \frac{d}{b}$.

$\frac{a}{c} = \frac{b}{d}$.

$\frac{a}{c} + 1 = \frac{b}{d} + 1$.

$\frac{a+c}{c} = \frac{b+d}{d}$.

$\frac{c}{a+c} = \frac{d}{b+d}$.

f. $\frac{c}{a} = \frac{d}{b}$.

$\frac{c}{a} - 1 = \frac{d}{b} - 1$.

$\frac{c-a}{a} = \frac{d-b}{b}$.

4. a. Each fraction is inverted.

b. The denominator of the first fraction and the numerator of the second have exchanged places.

c. We have now, the sum of the two terms of one fraction is to the numerator of that fraction, as the sum of the two terms in the second fraction is to the numerator of the second fraction.

d. Here in each fraction we have the difference of numerator and denominator divided by the numerator.

5. 1. c.   2. a.   3. f.   4. e.

6. a. $\frac{a}{x} = \frac{2}{3}$ b. $\frac{4}{3} = \frac{5}{m}$.

c. $\frac{4}{7} = \frac{b}{a}$ or $\frac{4}{b} = \frac{7}{a}$.   d. $\frac{5}{5} = \frac{x}{9}$.

7. a. $AB = \frac{AC \cdot DE}{DF}$. b. $BC = \frac{AB \cdot EF}{DE}$.

c. $AC = \frac{BC \cdot DF}{EF}$. d. $AB = \frac{DE \cdot BC}{EF}$.

e. $BC = \frac{AC \cdot EF}{DF}$. f. $AC = \frac{DF \cdot AB}{DE}$.

8. a. $AE = 2$.

b. $DE \parallel BC$.

9. a. G.M. = 6,(6.000); A.M. = 6.5.

b. G.M. = $6\sqrt{2}$, (8.484); A.M. = 9.0.

c. G.M. = $4\sqrt{5}$, (8.944); A.M. = 9.0.

d. G.M. = $4\sqrt{3}$, (6.928); A.M. = 13.0.

e. G.M. = $\sqrt{6}$, (2.449); A.M. = 2.5.

Problem Set 12-4a.

1. a. $AB = 5\frac{5}{7}$. b. $BF = 5$. c. $BF = 13\frac{1}{2}$.

2. a. $BC = 24$.

b. $CE = 6\frac{2}{3}$.

c. $AC = 11$.

d. $BE = 7\frac{1}{2}$.

e. $AD = 10$.

3. No. $\frac{20}{16} \neq \frac{30}{23}$.

4. a, b, e.

5. a. By Theorem 12-1, $\frac{CA}{CD} = \frac{CB}{CF}$.

Then

$$\frac{CA}{CD} - 1 = \frac{CB}{CF} - 1.$$

or

$$\frac{CA-CD}{CD} = \frac{CB-CF}{CF}.$$

Therefore

$$\frac{DA}{CD} = \frac{FB}{CF}.$$

b. Taking the reciprocals of both fractions of (a) we get

$$\frac{CD}{DA} = \frac{CF}{FB}.$$

Then

$$\frac{CD}{DA} + 1 = \frac{CF}{FB} + 1$$

or

$$\frac{CD+DA}{DA} = \frac{CF+FB}{FB}.$$

Therefore,

$$\frac{CA}{DA} = \frac{CB}{FB}.$$

c. By Theorem 12-1, $\frac{CA}{CD} = \frac{CB}{CF}$.

Clearing of fractions, $CA \cdot CF - CD \cdot CB$,

and dividing by $CF \cdot CB$ we have

$$\frac{CA}{CB} = \frac{CD}{CF}.$$

6. $\frac{w}{19} = \frac{9}{16}$ is one. $w = \frac{171}{16}$.

7. x must be 8 or 11.

8. Proof:

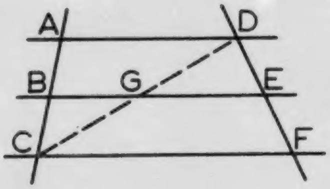
Statements	Reasons
1. $\overline{EF} \parallel \overline{AB}$. $\overline{FG} \parallel \overline{BC}$. $\overline{GH} \parallel \overline{DC}$	1. Given.
2. $\frac{XA}{XE} = \frac{XB}{XF}$. $\frac{XB}{XF} = \frac{XC}{XG}$. $\frac{XC}{XG} = \frac{XD}{XH}$.	2. Theorem 12-1.

3. $\frac{XA}{XE} = \frac{XD}{XH}$. 3. From step 2.

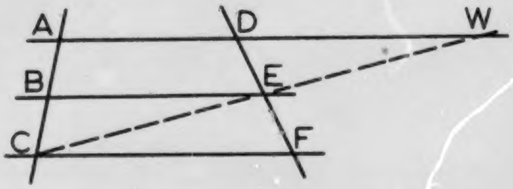
4. $\overline{HE} \parallel \overline{AD}$. 4. Theorem 12-2.

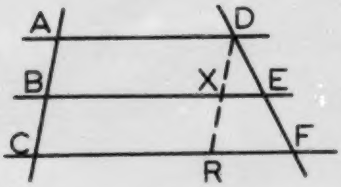
No, the figure does not have to be planar.

9. Proof: Draw transversal $\overleftrightarrow{DC}$ intersecting $\overleftrightarrow{BE}$ in G. In $\triangle CAD$ we have by Theorem 12-1, $\frac{AC}{BC} = \frac{CD}{CG}$ from which $\frac{AB}{BC} = \frac{DG}{GC}$. Similarly, in $\triangle DCF$, we get $\frac{DG}{GC} = \frac{DE}{EF}$. Hence, $\frac{AD}{BC} = \frac{DE}{EF}$.



(An alternate method of proof might use an auxiliary line $\overleftrightarrow{CW}$ as shown at the right, or a line $\overleftrightarrow{DR} \parallel \overleftrightarrow{AC}$ as shown here.





10. Lot r: 80 ft. Lot II: 160 ft. Lot III: 120 ft.

11. Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{XY}$,

$$\frac{OA}{OX} = \frac{OB}{OY}.$$

Similarly $\overleftrightarrow{BC} \parallel \overleftrightarrow{YZ}$ implies

$$\frac{OB}{OY} = \frac{OC}{OZ}.$$

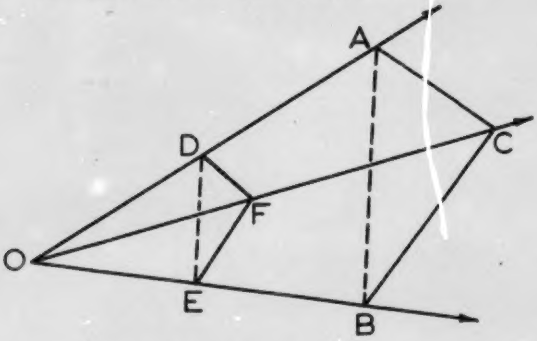
Hence $\frac{OA}{OX} = \frac{OC}{OZ}$. This implies $\overleftrightarrow{AC} \parallel \overleftrightarrow{XZ}$ by 11-2.

12. x will be the length of the folded card,

    So $\frac{6}{x} = \frac{x}{3}$ and $x^2 = 18$.

    The width of the card should be $\sqrt{18}$ or $3\sqrt{2}$ inches.

13. No. We can be sure that it is when the plane of the triangle and the plane of the film are parallel.



Proof: Assuming that the planes of $\triangle ABC$ and $\triangle DEF$ are parallel, $\overline{DE} \parallel \overline{AB}$, $\overline{EF} \parallel \overline{BC}$, $\overline{DF} \parallel \overline{AC}$.

$$\therefore \triangle ODE \sim \triangle OAB, \quad \triangle OEF \sim \triangle OBC, \quad \triangle OFD \sim \triangle OCA.$$

$$\frac{EF}{BC} = \frac{OE}{OB} = \frac{ED}{BA} = \frac{OD}{OA} = \frac{DF}{AC}\text{, that is,}$$

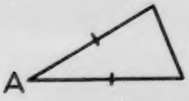
$$\frac{EF}{BC} = \frac{ED}{BA} = \frac{DF}{AC}.$$

Therefore $\triangle ABC \sim \triangle DEF$ by S.S.S. Similarity.

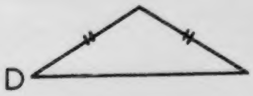
Problem Set 12-4b.

1. Similarities are indicated in a, c, d.

   Notice that the wording of (e) permits



and



2. The A.A.A. and the A.A. theorems.

3. a. No. b. Yes. c. No. d. Yes.

4. a. $\angle AXC$, or $\angle BXC$.

   b. $\angle ACX$.

c. $\triangle AXC$, or $\triangle CXB$.

5. $XC = \frac{qr}{p}$, or $XG = \frac{qp}{r}$. No.

6. a. $\triangle ABF \sim \triangle QRS$. $\frac{AB}{QR} = \frac{AF}{QS} = \frac{BF}{RS} = \frac{1}{3}$.

b. $\triangle MTW \sim \triangle RLS$. $\frac{MT}{RL} = \frac{MW}{RS} = \frac{TW}{LS} = \frac{2}{3}$.

c. $\triangle ABC$ is not $\sim \triangle XYZ$.

d. $\triangle ABC \sim \triangle TSR$. $\frac{AB}{TS} = \frac{AC}{TR} = \frac{BC}{SR} = \frac{1}{5}$.

e. $\triangle ABC \sim \triangle TWX$. $\frac{AB}{TW} = \frac{BC}{WX} = \frac{AC}{TX} = 6$.

Proof: $\triangle ABC \sim \triangle CDL$ since the vertical angles at L are congruent as well as the given angles B and D. From the given information $\frac{CD}{AB} = \frac{4}{1}$. Since the triangles have been proved similar $\frac{DL}{BL} = \frac{4}{1}$. Then $\frac{DL + BL}{BL} = \frac{4 + 1}{1}$.

7. Since L is between B and D, this can be written

$\frac{BD}{BL} = \frac{5}{1}$ or $BD = 5BL$.

8. a. $\frac{r}{1} = \frac{s}{x}$, $rx = s$, $x = \frac{s}{r}$.

b. $\frac{1}{m} = \frac{p}{x}$, $x = mp$.

c. $\frac{1}{k} = \frac{k}{x}$, $x = k^2$.

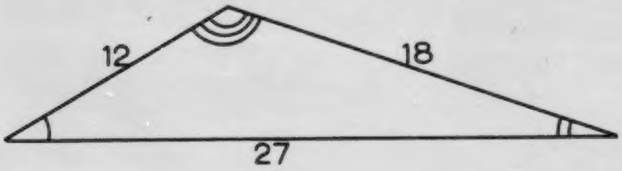
d. $\frac{t}{1} = \frac{1}{x}$, $xt = 1$, $x = \frac{1}{t}$.

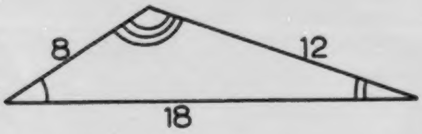
e. Part b.

f. Part a.

g. No.

9. Of the five equal pairs of parts three must be angles, for if three were sides the triangles would be congruent. Hence the triangles are similar. Neither of the two pairs of equal sides can be corresponding sides or the triangles would be congruent by A.S.A. The remaining possibility can best be shown by an example.





10. Proof: 1. $\triangle OBX \sim \triangle O_1B_1x$ by A.A.A.

2. Therefore $\frac{OB}{O_1B_1} = \frac{OX}{O_1X}$.

3. $\triangle ODX \sim \triangle O_1D_1X$ by A.A.A.

4. Therefore $\frac{OD}{O_1D_1} = \frac{OX}{O_1X}$.

5. From statements 2 and 3, $\frac{OB}{O_1B_1} = \frac{OD}{O_1D_1}$.

11. a. The triangles are similar. S.S.S.

b. Not similar.

c. The triangles are similar. A.A.A. or S.A.S.

d. Similar. A.A.A.

e. Similar. S.S.S.

f. Similar. A.A. or S.A.S.

12. a. $\triangle BSC \sim \triangle BTD$, $\triangle DSC \sim \triangle DRB$, $\triangle RSB \sim \triangle DST$.

b. $\frac{z}{y} = \frac{p}{p+q}$.

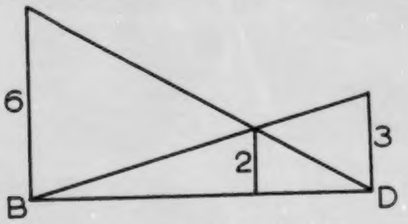
c. $\frac{z}{x} = \frac{q}{p+q}$.

d. $\frac{z}{x} + \frac{z}{y} = \frac{p+q}{p+q}$.

$$\frac{z}{x} + \frac{z}{y} = 1.$$

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}.$$

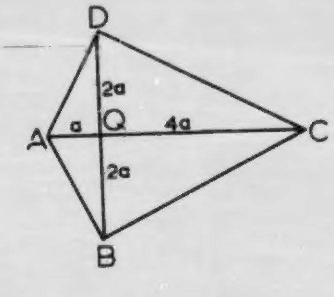
e. Construct perpendiculars 6 and 3 units long at opposite ends (but on the same side) of any segment BD. Join the ends of these perpendiculars to the opposite ends of the segment, and where these lines intersect draw a perpendicular to BD. Measure this perpendicular. It should be 2 units long. Therefore the task would require 2 hours.



13. Proof:

Statements	Reasons
1. ABRQ is a parallelogram.	1. Given.
2. $\angle QHA \cong \angle BHF$.	2. Vertical angles.
3. $\overline{AQ} \parallel \overline{RB}$	3. Opposite sides of a parallelogram are parallel.
4. $\angle AQB \cong \angle FBH$.	4. Alternate interior angles.
5. $\triangle AHQ \sim \triangle FHB$.	5. A.A.
6. $\frac{AH}{FH} = \frac{HQ}{HB}$.	6. Definition of similar triangles.
7. $AH \cdot HB = FH \cdot HQ$.	7. Clearing of fractions.

14. Proof: a. and b. Let a, 2a, 4a stand for the lengths as shown in the figure. Then it can easily be shown for each pair of triangles mentioned that



the S.A.S. Similarity Theorem applies.

c. $\angle AQD$ and $\angle DQC$ are corresponding angles of similar triangles and therefore are congruent.

15. Let $\overleftrightarrow{BE}$ be parallel to $\overleftrightarrow{AD}$, meeting $\overleftrightarrow{AC}$ in E.

$\angle ABE \cong \angle DAB$ (alt. int. $\angle$s) and $\angle AEB \cong \angle CAD$ (corr. $\angle$s).

Also, $\angle DAB \cong \angle CAD$ (given). Therefore $\angle AEB \cong \angle ABE$.

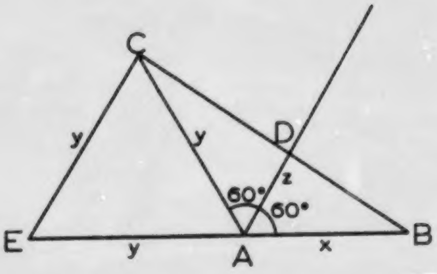
Therefore AE = AB. Since $\frac{CD}{DB} = \frac{CA}{AE}$, then $\frac{CD}{DB} = \frac{CA}{AB}$ by substitution.

16. From the previous problem $\frac{CD}{DB} = \frac{CA}{AB}$.

By an exactly similar proof you can show that $\frac{CD'}{D'B} = \frac{CA}{AB}$.

Therefore $\frac{CD'}{D'B} = \frac{CD}{DB}$.

17. a. Let E be the point on the ray opposite to $\overrightarrow{AB}$ such that AE = y. Then $\triangle AEC$ is equilateral, EC = y. In the similar triangles ECB and ADB,



$\frac{EC}{AD} = \frac{EB}{AB}$, or

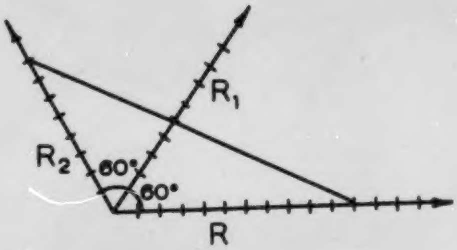
$\frac{y}{z} = \frac{x+y}{x}$,

$\frac{y}{z} = 1 + \frac{y}{x}$.

Dividing by y, we get

$\frac{1}{z} = \frac{1}{y} = \frac{1}{x}$.

b. Yes, place the straight-edge against R_1 on the middle scale and R_2 on one of the outer scales. Then read off R on the other outer scale.



18. Proof:

Statements	Reasons
1. $\frac{RW}{AL} = \frac{WS}{LQ} = \frac{RT}{AM}$.	1. Given.
2. $\frac{RT}{AM} = \frac{\frac{1}{2}RT}{\frac{1}{2}AM} = \frac{RS}{AQ}$.	2. Given $\overline{WS}$ and $\overline{LQ}$ are medians.
3. $\frac{KW}{AL} = \frac{WS}{LQ} = \frac{RS}{AQ}$.	3. Steps 1 and 2 and substitution.
4. $\triangle RSW \sim \triangle AQL$.	4. S.S.S. similarity.
5. $\angle R \cong \angle A$.	5. Definition of similar triangles.
6. $\triangle RWT \sim \triangle ALM$.	6. Step 1 and Theorem 12-4.

19. Proof:

Statements	Reasons
1. $\angle y$ is the complement of $\angle x$.	1. $\overline{RA} \perp \overline{AB}$, and definition of complementary angles.
2. $\angle y$ is the complement of $\angle R$.	2. Given $RH \perp AF$, and acute angles of a right triangle are complementary.
3. $\angle x \cong \angle R$.	3. Complements of the same angle are congruent.
4. $\angle B \cong \angle RHA$.	4. $RH \perp AF$ and $FB \perp AB$.
5. $\triangle HRA \sim \triangle BAF$.	5. Two angles of one congruent. to two angles of the other. (A.A.)

6. $\frac{HR}{BA} = \frac{HA}{BF}$.	6. Definition of similar triangles.
7. $HR \cdot BF = BA \cdot HA$.	7. Clearing of fractions in Step 6.

20. a. No.

b. Bisect PA_1, PB_1 etc., and connect the resulting midpoints.

c. $\frac{PA_2}{PA_1} = \frac{PB_2}{PB_1}$ because both equal 2. $\angle A_1PB_1$ is common to triangles A_1PB_1 and A_2PB_2. These triangles are therefore similar by the S.A.S. similarity theorem; and as a result of their being similar the sides A_2B_2 and A_1B_1 have the same ratio as the other corresponding sides.

d. Not only A_2B_2 and A_1B_1 but other corresponding sides of triangle, $A_2B_2D_2$ and $A_1B_1D_1$ are in 2:1 ratio by a proof like that in part c. $\triangle A_2B_2D_2 \sim \triangle A_1B_1D_1$ by the S.S.S. similarity theorem.

e. Yes, the method could be used for any point P; but in some instances the enlargement would intersect the given figure.

21. $\angle SRX \cong \angle QTX$ and

$\angle RSX \cong \angle TQX$ (alternate interior $\angle$s), so $\triangle SRX \sim \triangle QTX$ by A.A. $\therefore \frac{RX}{TX} = \frac{SX}{QX}$, so $\frac{RX}{SX} = \frac{TX}{QX}$. Since $\triangle QXR \sim \triangle TXS$ (given), $\frac{RX}{SX} = \frac{QX}{TX}$. $\therefore \frac{QX}{TX} = \frac{TX}{QX}$, $(QX)^2 = (TX)^2$, and $QX = TX$, since both QX and TX are positive.

$\angle XQR \cong \angle XTS$ and $\angle RXQ \cong \angle SXT$ (similar $\triangle$s), so $\triangle QXR \sim \triangle TXS$ by A.S.A. $\therefore QR = TS$.

Alternate proof. If $TS > QR$, then $TX > QX$ and $XS > XR$, from $\triangle QXR \sim \triangle TXS$. In $\triangle QXT$, $m\angle XQT > m\angle XTQ$, by Th. 7-2, and in $\triangle RXS$, $m\angle SRX > m\angle RSX$. But $m\angle XTQ = m\angle SRX$, by Alt. Int. $\angle$s, and $m\angle XQT = m\angle RSX$. Contradiction. Similarly if $QR > TS$.

22. Proof:

Statements	Reasons
1. $\overline{AW} \perp \overline{MW}$. BFRQ is a square.	1. Given.
2. $\angle ABQ \cong \angle W \cong \angle MFR$.	2. Definitions of "perpendicular" and "square".
3. Let the measure of $\angle A$ be a and the measure of $\angle M$ be m.	3. Angle measurement postulate.
4. Thus, $m\angle FRM = a$ and $m\angle AQB = m$.	4. The acute angles of a right triangle are complementary.
5. Also, $m\angle WQR = a$ and $m\angle WRQ = m$.	5. The sum of the measures of the angles at Q is 180 and the sum of the measures of the angles at R is 180.
6. $\triangle ABQ \sim \triangle RFM \sim \triangle QWR$.	6. A.A.A.
7. $\frac{AB}{QW} = \frac{BQ}{WR}$ and $\frac{AB}{RF} = \frac{BQ}{FM}$.	7. Definition of similar triangles.
8. $AB \cdot WR = QW \cdot BQ$ and	8. Clearing of fractions in Step 7.

23. Proof: Since $\triangle ABF \sim \triangle HRQ$ we know $\angle F \cong \angle Q$ and

$$\frac{AF}{HQ} = \frac{AB}{HR} = \frac{BF}{RQ}. \quad \text{Also } \frac{FB}{QR} = \frac{\frac{1}{2}FB}{\frac{1}{2}QR} = \frac{FW}{QX} = \frac{AF}{QH}. \quad \text{Then}$$

$\triangle AWF \sim \triangle HXQ$ by S.A.S. Similarity, and then

$$\frac{AW}{HX} = \frac{AF}{HQ} = \frac{FB}{QR} = \frac{AB}{HR}.$$

It is possible to continue in the same way for the other medians.

24. Proof: Since $\triangle ABF \sim \triangle XWR$ then $\angle x \cong \angle A$ and

$\frac{XR}{AF} = \frac{XW}{AB} = \frac{WR}{BF}$. $m\angle AHF = m\angle XQR$ and so

$\triangle XQR \sim \triangle AHF$ by A.A. Then $\frac{RQ}{FH} = \frac{XR}{AF}$.

A similar proof can be followed for each of the altitudes.

25. Proof: As shown in the two figures the two triangles are similar by A.A.

26. Solution: Since the base angles are congruent $AE = BE$ and by subtraction $EC = ED$. Hence $\triangle CED \sim \triangle AEB$ by the S.A.S. Similarity Theorem.

Therefore $\angle ECD \cong \angle EAB$ and $\overline{CD} \parallel \overline{AB}$ by corresponding angles.

27. False. Let $\triangle AB_1C$ and $\triangle AB_2C$ be such that $AC = AC$, $\angle A \cong \angle A$, $CB_1 = CB_2$, as in the diagram, but the triangles are not congruent. Construct $\triangle A'B'C' \sim \triangle AB_1C$. The triangles $A'B'C'$ and AB_2C satisfy the statements of the hypotnesis, but these triangles are not similar.

28. a. 1. $\triangle ABC \sim \triangle ADE$; $\frac{AB}{AD} = \frac{AC}{AE} = \frac{BC}{DE}$.

2. $\triangle ABC$ and $\triangle ADF$ are not similar even though

$\frac{AB}{AD} = \frac{BC}{DF}$ since $m\angle B \neq m\angle FDA$.

b. False. The diagram shows a counter-example. The hypothesis is true if X is either E or F. The conclusion is false if X is F.

29. In similar $\triangle$ABC and EDC, $\frac{x}{39} = \frac{a}{b}$.

From the similar $\triangle$ ACG and AEF,

$\frac{a+b}{a} = \frac{7}{3}$,

$1 + \frac{b}{a} = \frac{7}{3}$,

$\frac{b}{a} = \frac{4}{3}$,

$\frac{a}{b} = \frac{3}{4}$,

$\frac{x}{39} = \frac{3}{4}$,

$x = \frac{3}{4} \cdot 39 = 29\frac{1}{4}$.

Answer. The ball hits the ground at least 29'3" from the net.

30. $\triangle CEB \sim \triangle AEF$ since $\angle x \cong \angle y$ (alternate interior angles of parallel lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ and $\angle FEA \cong \angle BEC$ (vertical angles); therefore $\frac{EF}{EB} = \frac{FA}{BC} = \frac{AE}{CE}$. Also, $\triangle CEG \sim \triangle AEB$ since $\angle ABE \cong \angle CGE$ (alternate interior angles) and $\angle CEG \cong \angle AEB$ (vertical angles), we get $\frac{BA}{GC} = \frac{AE}{CE} = \frac{EB}{EG}$. Since in each case we have $\frac{AE}{CE}$ as one of the fractions, we also have $\frac{EF}{EB} = \frac{EB}{EG}$.

31. Proof: Since $\overleftrightarrow{AX} \parallel \overleftrightarrow{BY}$, $\triangle DAX \sim \triangle DBY$ and $\frac{DA}{DB} = \frac{AX}{BY}$.

Similarly since $\overleftrightarrow{CZ} \parallel \overleftrightarrow{BY}$, $\triangle CEZ \sim \triangle BEY$ and $\frac{EC}{EB} = \frac{CZ}{BY}$.

But $AX = CZ$, since opposite sides of a parallelogram are congruent and so $\frac{DA}{DB} = \frac{EC}{EB}$. Now $1 - \frac{DA}{DB} = 1 - \frac{EC}{EB}$, $\frac{DB - DA}{DB} = \frac{EB - EC}{EB}$ and $\frac{AB}{DB} = \frac{BC}{EB}$. Therefore $AC \parallel DE$ by Theorem 11-2. And now $\overleftrightarrow{AC} \parallel \overleftrightarrow{DE} \parallel \overleftrightarrow{XZ}$.

Problem Set 12-6.

1. $x = 2\sqrt{5}$.

   $z = 6$.

   $y = 3\sqrt{5}$.

2. $x = 16$.

   $y = 4\sqrt{5}$.

   $z = 8\sqrt{5}$.

3. a. $\frac{4}{6} = \frac{6}{x+4}$.

   $36 = 4x + 16$.

   $20 = 4x$.

   $5 = x$.

   b. $\frac{4}{y} = \frac{y}{5}$.

   $y^2 = 4 \cdot 5$.

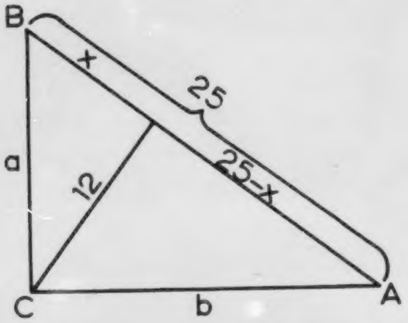
   $y = 2\sqrt{5}$.

   c. $\frac{9}{a} = \frac{a}{5}$.

   $a^2 = 9 \cdot 5$.

   $a = 3\sqrt{5}$.

4. Let the segments of the hypotenuse be x and $25 - x$. Then $\frac{x}{12} = \frac{12}{25-x}$ by Theorem 11-6 and definition of similar triangles.



   $144 = 25x - x^2$.

   $x^2 - 25x + 144 = 0$.

   $(x-9)(x-16) = 0$. The segments of the hypotenuse are 9 and 16.

   If a is the length of the shorter leg,

   $\frac{25}{a} = \frac{a}{9}$

   $a = 15$.

   $\frac{25}{b} = \frac{b}{16}$,

   $b = 20$.

5. By Theorem 12-6, $\triangle ABC \sim \triangle CBF$ and $\frac{AB}{CB} = \frac{CB}{BF}$. From this we get $CB = \sqrt{AB \cdot BF}$. Also, $\triangle ABC \sim \triangle ACF$ and $\frac{AB}{AC} = \frac{AC}{AF}$. Then $AC = \sqrt{AB \cdot AF}$.

6. Given: $\triangle ABC$, $\angle C$ a right angle and $\overline{CH} \perp \overline{AB}$.

   Prove: $CH = \sqrt{AH \cdot HB}$.

   Proof: $\triangle AHC \sim \triangle CHB$, by Theorem 12-6.

   Therefore, $\frac{AH}{HC} = \frac{CH}{HB}$.

   $$(CH)^2 = AH \cdot HB.$$

   $$CH = \sqrt{AH \cdot HB}.$$

7. a. $CD = 4$; $AC = \sqrt{20} = 2\sqrt{5}$; $CB = \sqrt{80} = 4\sqrt{5}$.

   b. $DB = 27$; $AC = \sqrt{90} = 3\sqrt{10}$; $CB = \sqrt{810} = 9\sqrt{10}$.

   c. Let $DB = x$, then $x(x+10) = 144$.

   $$x^2 + 10x = 144.$$

   $$x^2 + 10x - 144 = 0.$$

   $$(x+18)(x-8) = 0.$$

   $$x = 8.$$

   $DB = 8$.

   $CA = \sqrt{180} = 6\sqrt{5}$.

   $CD = \sqrt{80} = 4\sqrt{5}$.

   d. Let $AD = x$, then

   $x(x+12) = 64$. $\qquad AD = 4$.

   $x^2 + 12x - 64 = 0$. $\qquad CB = 8\sqrt{3}$.

   $(x-4)(x+16) = 0$. $\qquad CD = \sqrt{48} = 4\sqrt{3}$.

   $x = 4$.

8. $\frac{9}{16}$; $\frac{x^2}{y^2}$.

9. $(\frac{b}{20})^2 = \frac{36}{225} = (\frac{6}{15})^2 = (\frac{2}{5})^2$,

$\frac{b}{20} = \frac{2}{5}$, $b = \frac{40}{5} = 8$.

The base of the smaller is 8 inches.

10. Since $\overline{DE} \parallel \overline{AB}$,

$\triangle ABC \sim \triangle DEC$.

$\frac{CA}{CD} = 3$ and so $\frac{\text{Area } \triangle ABC}{\text{Area } \triangle DEC} = 9$.

11. a. 2. b. 4.

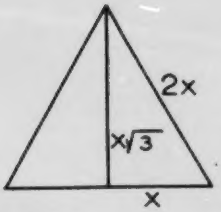
12. $(\frac{S}{10})^2 = \frac{2}{1}$.

$\frac{S^2}{100} = \frac{2}{1}$.

$S^2 = 100 \cdot 2$.

$S = 10\sqrt{2}$.

The sides will be $10\sqrt{2}$ inches.



13. $\frac{A_1}{A_2} = (\frac{2x}{x\sqrt{3}})^2 = \frac{4}{3}$.

14. If the length of the wire is called d, the side of the square is $\frac{1}{4}d$ and that of the triangle is $\frac{1}{3}d$. Then the area of the square is $\frac{d^2}{16}$ and that of the triangle is $\frac{d^2}{36}\sqrt{3}$. Then,

$$\frac{\text{Area of the triangle}}{\text{Area of the square}} = \frac{\frac{d^2}{36}\sqrt{3}}{\frac{d^2}{16}} = \frac{4\sqrt{3}}{9} .$$

15. The area of $\triangle ABC = \frac{1}{2}\, 140\cdot 120 = 8400$.

The area of the required lot must then be 4200. By the Pythagorean Theorem, $AD = 90$, and area of $\triangle ADC = \frac{1}{2}\cdot 90\cdot 120 = 5400$. Then, by Theorem 12-7, $\left(\frac{x}{90}\right)^2 = \frac{4200}{5400}$, and $x = 30\sqrt{7}$. The required distance is approximately 79.4 ft.

16. Given: Right $\triangle ABC$, $\angle C$ a right angle, and M the midpoint of $\overline{AB}$.

Prove: $MA = MB = MC$.

Proof: Let $\overline{MK}$ be the perpendicular from M to $\overline{BC}$, meeting $\overline{BC}$ in K. Then $\overline{MK} \parallel \overline{AC}$, so $CK = KB$. Therefore $\overleftrightarrow{MK}$ is the perpendicular bisector of $\overline{CB}$. Hence $MC = MB$. Since $MB = MA$ (given), then $MA = MB = MC$.

17. By Problem 16, $KC = \frac{c}{2}$, where $AB = c$. Therefore $m\angle KCB = m\angle KBC = 60$, so $m\angle BKC = 60$. Therefore $BC = KB = \frac{c}{2}$.

18. Since $AR = RC$,

$m\angle A = m\angle ACR$.

Also, since $RC = RB$,

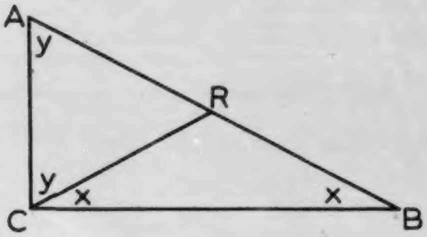
$m\angle B = m\angle BCR$.

Let $m\angle A = m\angle ACR = y$ and

$m\angle B = m\angle BCR = x$.

Then in $\triangle ACB$, $2x + 2y = 180$, and

$x + y = 90$.



19. $HC = \sqrt{AH\cdot HB}$, $(HC)^2 = AH\cdot HB$, $\frac{AH}{HC} = \frac{CH}{HB}$.

Also $\angle AHC \cong \angle CHB$.

Hence $\triangle AHC \sim \triangle CHB$ by S.A.S. Similarity Theorem. Therefore $\angle HCB \cong \angle A$. Since $\angle HCB$ and $\angle B$ are complementary, then

$\angle A$ and $\angle B$ are complementary, and $\triangle ACB$ is a right triangle. By the preceding problem MC = AM, and $MC = \frac{1}{2}AB = \frac{1}{2}(AH + HB)$. But HC < MC, except when M = H (i.e., when AH = HB). Therefore, $\sqrt{AH \cdot HB} = HC < \frac{1}{2}(AH + HB)$. If AH = HB, the last inequality becomes the equality $\sqrt{(AH)^2} = \frac{1}{2}(AH + AH)$, that is, AH = AH.

Alternate solution. Let u and v be positive numbers, $u \neq v$. Then

$$0 < (\sqrt{u} - \sqrt{v}) = u - 2\sqrt{u}\sqrt{v} + v.$$

$$2\sqrt{uv} < u + v.$$

$$\sqrt{uv} < \frac{u + v}{2}.$$

20. Outline of proof. $\triangle PXR \sim \triangle PYA$, $\therefore \frac{PR}{PA} = \frac{PX}{PY}$.

$$\triangle PRS \sim \triangle PAB, \therefore \frac{PR}{PA} = \frac{RS}{AB}.$$

$$\triangle RST \sim \triangle ABC, \therefore$$

$$\frac{\text{Area } \triangle RST}{\text{Area } \triangle ABC} = \left(\frac{RS}{AB}\right)^2.$$

From the above:

$$\frac{\text{Area } \triangle RST}{\text{Area } \triangle ABC} = \left(\frac{PX}{PY}\right)^2.$$

21. 1. Area Addition Postulate (Post. 19).
    2. Division
    3. Th. 12-6.
    4. Th. 12-7.
    5. Multiplication.

22. $a^2 = h^2 + y^2 = h^2 + (c - x)^2.$

$= (h^2 + x^2) + c^2 - 2cx.$

$= b^2 + c^2 - 2cx.$

In the similarity $\triangle ADC \sim \triangle RST$,

$$\frac{x}{b} = \frac{k}{1} = k,$$

$$x = bk.$$

Therefore

$$a^2 = b^2 + c^2 - 2bck.$$

23. $a^2 = h^2 + y^2 = h^2 + (x + c)^2.$

$= (h^2 + x^2) + c^2 + 2cx.$

$= b^2 + c^2 + 2cx.$

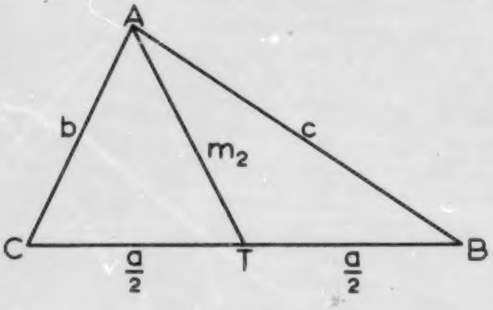
In the similarity $\triangle ADC \sim \triangle RST$,

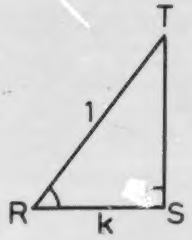
$\frac{x}{b} = \frac{k}{1} = k,$

$x = bk.$

Therefore $a^2 = b^2 + c^2 + 2bck.$

24. (a)





(This is the case in which $\angle C$ is acute. If $\angle C$ is obtuse or a right angle, the proof is similar.)

Let $\triangle RST$ have $\angle R \cong \angle C$, $\angle S$ a right angle, hypotenuse = 1, RS = k. By the result of EX. 16, applied to $\triangle ACT$,

$$m_a^2 = b^2 + \left(\frac{a}{2}\right)^2 - 2b\left(\frac{a}{2}\right)k,$$

(1) $$m_a^2 = b^2 + \frac{a^2}{4} - abk.$$

Applying the same result to $\triangle ACB$,

(2) $$c^2 = b^2 + a^2 - 2abk.$$

Multiplying both sides of equation (2) by $\frac{1}{2}$ and subtracting from the corresponding sides of equation (1):

$$m_a^2 - \frac{1}{2}c^2 = \frac{b^2}{2} - \frac{a^2}{4},$$

$$m_a^2 = \frac{1}{2}\left(b^2 + c^2 - \frac{a^2}{2}\right).$$

(b) From part (a) $$m_a^2 = \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{4}a^2,$$

$$m_b^2 = \frac{1}{2}a^2 + \frac{1}{2}c^2 - \frac{1}{4}b,$$

$$m_c^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{4}c^2.$$

Adding and collecting like terms,

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2).$$

Review: Chapters 7-12.

Solutions:

1. 0.
2. 1.
3. 1.
4. 0.
5. 1.
6. 0.
7. 0.
8. 0.
9. 1.
10. 1.
11. 1.
12. 1.
13. 1.
14. 0.
15. 1.
16. 1.
17. 0.
18. 0.
19. 1.
20. 0.
21. 1.
22. 1.
23. 0.
24. 1.
25 1.
26. 0.
27. 1.
28. 1.
29. 0.
30. 0.
31. 0.
32. 0.
33. 1.
34. 1.
35. 1.
36. 0.
37. 1.
38. 0.
39. 1.
40. 0.
41. 1.
42. 0.
43. 1.
44. 0.
45. 0.
46. 1.
47. 1.
48. 0.
49. 0.
50. 0.
51. 1.
52. 1.
53. 1.
54. 1.
55. 0.
56. 1.
57. 1.
58. 0.
59. 0.
60. 0.

Solutions of Problems

Problem Set 13-1.

1. a. False.
   b. True.
   c. False.
   d. False.
   e. False.
   f. True
   g. False.
   h. True.

2. a. False
   b. True.
   c. True.
   d. False.
   e. False.
   f. False.
   g. True.
   h. False.

3. Let m be the radius of any circle M and n be the radius of any circle N. Then the situations are:

   a. $m + n < MN$.

   b. $m > MN + n$ or $n > MN + m$.

4. Proof: By definition of a circle, R and Q are equidistant from P. That is $PR = PQ$, or $\overline{PR} \cong \overline{PQ}$.

5. Let c be the length of any chord not a diameter. Draw radii to its endpoints. Then $2r > c$, by Theorem 7-5, the triangle inequality. But $2r$ is the length of the diameter. Hence the diameter is larger than any other chord.

Problem Set 13-2.

1. Proof:

Statements	Reasons
1. $\overline{AO} \cong \overline{OB}$.	1. Definition of a circle.
2. $\overleftrightarrow{OT} \perp \overleftrightarrow{CD}$.	2. Corollary 13-1-1.
3. $\overleftrightarrow{AC} \perp \overleftrightarrow{CD}$, $\overleftrightarrow{BD} \perp \overleftrightarrow{CD}$.	3. Given.
4. $\overleftrightarrow{AC} \parallel \overleftrightarrow{OT} \parallel \overleftrightarrow{BD}$.	4. Lines perpendicular to the same line are parallel.
5. $\overline{CT} \cong \overline{TD}$.	5. Theorem 9 - 25.

Proof:

Statements	Reasons
6. $m\angle CTO = m\angle DTO = 90$.	6. Perpendicular lines form right angles.
7. $\overline{OT} \cong \overline{OT}$.	7. Identity.
8. $\triangle CTO \cong \triangle DTO$.	8. S.A.S.
9. $\overline{CO} \cong \overline{DO}$.	9. Corresponding parts.

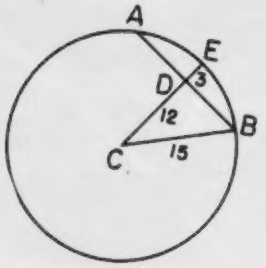
2. (See Teacher's Commentary.)

3. (See Teacher's Commentary.)

4. By Corollary 13-1-5, the perpendicular bisector of a chord passes through the center of the circle. Hence, to find the center draw any two chords in the circle and the perpendicular bisector of each. The intersection of these bisectors will be the center of the circle.

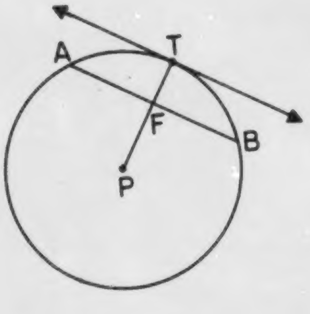
5. Draw a perpendicular from C to $\overline{MN}$, forming a 3-4-5 right triangle. Then the distance from C to $\overline{MN}$ is 16.

6. As in the figure, CB = 15 and DC = 12. Then DB = 9, and the chord is 18 inches long.



7. a. D.
   b. C.
   c. C.
   d. A.
   e. C.
   f. A.
   g. B.
   h. D.
   i. C.
   j. D.

8. Let $\overline{PT}$ intersect $\overline{AB}$ at F. Then FB = 6. $\triangle BFP$ is a $30^\circ - 60^\circ$ right triangle. Hence $PB = 4\sqrt{3}$.



9. Proof: Since a tangent to a circle is perpendicular to the radius drawn to the point of contact, the two tangents will be perpendicular to the same line and are, therefore, parallel.

10\.

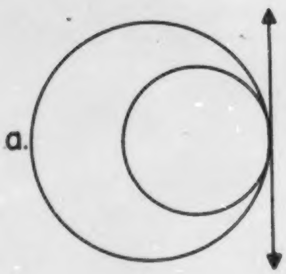
Statements	Reasons
1. $\overleftrightarrow{DO} \parallel \overleftrightarrow{AC}$. $\overleftrightarrow{CD}$ is tangent at C.	1. Given.
2. $\angle A \cong \angle BOD$.	2. Corresponding angles of parallels.
3. $OC = OA = OB$.	3. Definition of circle.
4. $\angle A \cong \angle ACO$.	4. Base angles of an isosceles triangle.
5. $\angle ACO \cong \angle COD$.	5. Alternate interior angles.
6. $\angle COD \cong \angle BOD$.	6. Steps 2,4,5.
7. $OD = OD$.	7. Identity.
8. $\triangle OCD \cong \triangle OBD$.	8. S.A.S and steps 3, 6, 7.
9. $\angle OCD \cong \angle OBD$.	9. Definition of congruent triangles.
10. $m \angle OCD = 90$.	10. Corollary 13-1-1.
11. $m \angle OBD = 90$.	11. Steps 9 and 10.
12. $\overleftrightarrow{DB}$ is tangent at B.	12. Corollary 13-1-5.

11. Proof: Draw $\overline{OR}$. $\overline{OR} \perp \overline{AB}$, by Corollary 13-1-1.

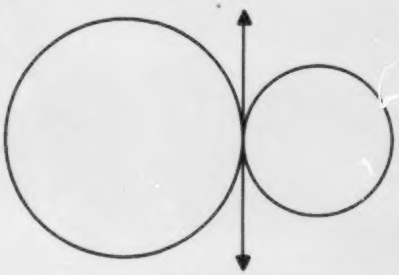
$AR = BR$, by Corollary 13-1-3.

12\.

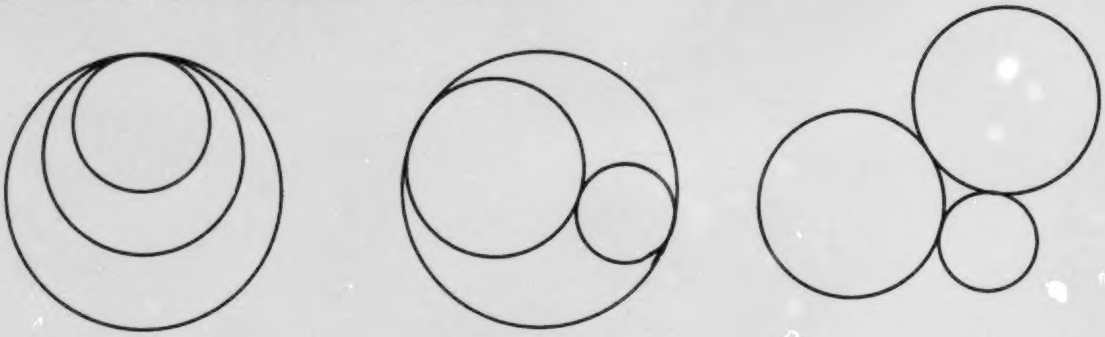
a.



b.



13. Here are three arrangements.

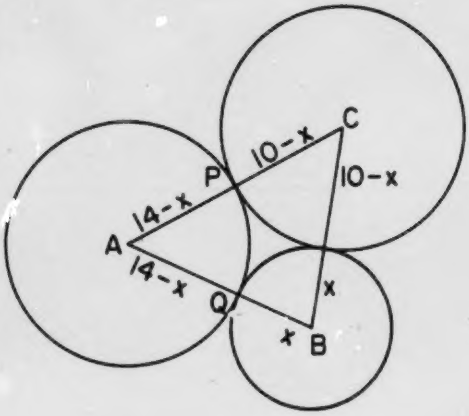


14. Let L be the common tangent. Then in both cases, $\overline{PT} \perp L$ and $\overline{QT} \perp L$ by Corollary 13-1-1. But there exists only one perpendicular to a line at a point on the line. Hence $\overline{PT}$ and $\overline{QT}$ are collinear. This means that P, Q, and T are collinear.

15. $AC = 14 - x + 10 - x = 18.$

$$24 - 2x = 18.$$
$$2x = 6.$$
$$x = 3.$$

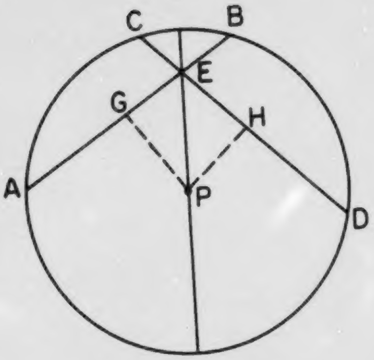
$BR = 3$, $CP = 7$, $AQ = 11$.



16. (See Teacher's Commentary.)

17. Given: $\angle AEP \cong \angle DEP$.

Prove: $\overline{AB} \cong \overline{CD}$.



Proof:

Draw $\overline{PG} \perp \overline{AB}$ and $\overline{PH} \perp \overline{CD}$. Then $\triangle PGE$ and $\triangle PHE$ are right triangles with m $\angle GEP$ = m $\angle HEP$, and EP = EP. Hence, $\triangle PGE \cong \triangle PHE$, making PG = PH. By Theorem 13-2, $\overline{AB} \cong \overline{CD}$.

18. Since PD = PE, AB = BC by Theorem 13-2. But DA = $\frac{1}{2}$ AB and EC = $\frac{1}{2}$ BC by Corollary 13-1-3. Hence, DA = EC.

19. Proof: By Corollary 13-1-4 the segment joining a midpoint of a chord to the center is perpendicular to the chord. By Theorem 13-3 these segments all have equal lengths. By the definition of a circle, all points equidistant from a point lie on the circle having the point as center and its radius equal to the distance. By Corollary 13-1-2 the chords are all tangent to the inner circle.

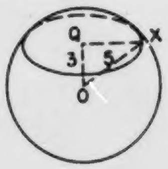
Problem Set 13-3.

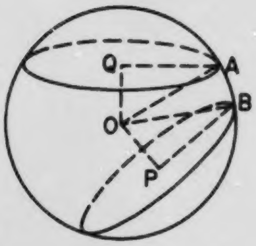
1. $\overleftrightarrow{OA} \perp \overleftrightarrow{FB}$.

   $\overleftrightarrow{OA} \perp \overleftrightarrow{RT}$.

2. By Corollary 13-4-3, the perpendicular bisects the chord. By Pythagorean Theorem, one-half the chord is 8, so the length of the chord is 16.

3. By the Pythagorean Theorem, QX = 4 inches.





4. $\overline{OQ}$ and $\overline{OP}$ are perpendicular to the planes of the circles. Therefore $\overline{OQ} \perp \overline{QA}$ and $\overline{OP} \perp \overline{PB}$. $OA = OB$, by the definition of sphere, and $OQ = OP$, by hypothesis. Then, by the Pythagorean Theorem, $QA = PB$. Hence, circle Q $\cong$ circle P be definition.

5. $AF = BF$ since they are radii of the circle of intersection, and $CF = AF$ by hypothesis. Also, $\overline{OF} \perp \overline{AF}$, $\overline{OF} \perp \overline{BF}$, and $\overline{AF} \perp \overline{BF}$. Hence, $\triangle AFB \cong \triangle AFO \cong \triangle BFO$, and $\triangle AOB$ is equilateral. Therefore $AO = 5$, $m \angle AOB = 60$, and OG, the altitude of $\triangle AOB$, equals $\frac{5}{2}\sqrt{3}$.

6. Call the three points A, B, C. To find the center of the circle, construct the perpendicular bisectors of any two of the three segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$. The bisectors intersect at the center, Q, of the circle. $\overline{QA}$, $\overline{QB}$, or $\overline{QC}$ is a radius of the circle.

   Construct the perpendicular to plane ABC at Q. This perpendicular meets the sphere in two points, X and Y. Determine the midpoint, P, of $\overline{XY}$. P is the center of the circle. $\overline{PA}$, $\overline{PB}$, or $\overline{PC}$ is a radius of the sphere.

7. By Theorem 13-4 we know that plane F intersects S in a circle. By Postulate 5, the two planes intersect in a line. Since both intersections contain T, the circle and line intersect at T. If they are not tangent at T, then they would intersect in some other point, R, also. Point R would then lie in plane E and in

sphere S. But this is impossible, since E and S are tangent at T. Hence, the circle and the line are tangent, by definition.

8. By definition, a great circle lies in a plane through the center of the sphere. The intersection of the two planes must contain the center of the sphere, so that the segment of the intersection which is a chord of the sphere is a diameter of the sphere, and also of each circle.

9. The plane of the perpendicular great circle is the plane perpendicular to the line of intersection of the planes of the given two, at the center of the sphere. There is only one such plane, by Theorem 8-11.

   Any two meridians have the equator as their common perpendicular.

10. The intersection of the spheres is a circle. This can be shown as follows: Let M and M' be any points of the intersection. Then $\triangle AMB \cong \triangle AM'B$ by S.S.S. If MO and M'O' are altitudes from M and M', $\triangle AMO \cong \triangle AM'O'$ by A.A.S., so that AO = AO' and O = O'. Hence all points M lie on a plane perpendicular to $\overleftrightarrow{AB}$ at O and on a circle with center O and radius OM.

    Since A and B are each equidistant from M and N, then all points on $\overleftrightarrow{AB}$ are equidistant from M and N, by Theorem 8-1, and $\overleftrightarrow{AB}$ is perpendicular to the plane of the intersection, by the argument above. By Theorem 11-11, we have MO = 5 in $\triangle$ MOB. In $\triangle$ MOA, by Pythagorean Theorem, we get AO = 12. But OB = 5. Hence AB = 17.

Problem Set 13-4a.

1. The center is the intersection of the perpendicular bisectors of two or more chords of the arc. (See Problem 13-2-4.)

2. Since an inscribed angle is measured by half the arc it intercepts, $\widehat{AB}$ must contain 90°. Since the measure of a central angle is the measure of its intercepted arc, $m \angle P = 90$ and $\overline{BP} \perp \overline{AP}$.

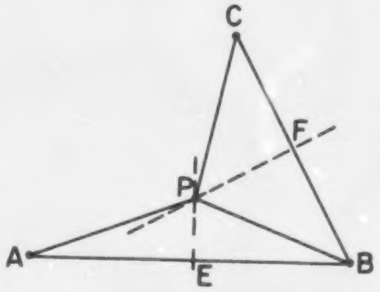
3. a. $m \angle A = m \angle B$ by Corollary 13-6-2.

   $m \angle AHK = m \angle BHF$ since the intercepted arcs have equal measure. Therefore $\triangle AHK \sim \triangle BHF$ by A.A. Similarity Theorem.

   b. $\triangle BFK$, since $m \angle BFA = \frac{1}{2} m \widehat{AB} = \frac{1}{2} m \widehat{BF} = m \angle BHF$, and $\angle HBF$ is common to the triangles.

4. Draw $\overline{RO}$. We know that $\overline{AO}$ is a diameter of the smaller circle and therefore that $m \angle ARO = 90$, by Corollary 13-6-1. Then $\overline{AB}$ is bisected by the smaller circle at point R, by Corollary 13-1-3.

5. Draw $\overline{AB}$ and $\overline{BC}$ and draw the perpendicular bisector of each segment. Since the segments $\overline{AB}$ and $\overline{BC}$ are not parallel or collinear, the perpendicular bisectors are not parallel and therefore intersect in a point P. This can be seen by using Theorem 9-11, Theorem 9-1, and the Parallel Postulate, in that order. $\triangle AEP \cong \triangle BEP$ by S.A.S., so that $AP = BP$, and $\triangle BFP \cong \triangle CFP$., by S.A.S. so that $BP = CP$. Hence $AP = BP = CP$. By definition of circle, A,B,C must lie on a circle with center P.



6. $m \angle C = \frac{1}{2} m \widehat{DAB}$.

$m \angle A + \frac{1}{2} m \widehat{DCB}$.

Since the sum of these two arcs is the entire circle, $m \angle C + m \angle A = 180$. Similarly, $m \angle B + m \angle D = 180$.

7. $m \widehat{ST} = 80$,

$m \widehat{RV} = 150$,

$m \angle T = 95$,

$m \angle V = 60$,

$m \angle S = 120$.

8. By the theorem of Problem 6 $\angle C$ and $\angle BXY$ are supplementary and $\angle D$ and $\angle AXY$ are supplementary. But $\angle AXY$ and $\angle BXY$ are supplementary. Therefore $\angle D$ and $\angle C$ are supplementary and so $AD \parallel BC$.

9. Draw radii $\overline{PA}$ and $\overline{PB}$. Since $CD \perp AB$, $AM = BM$ by Corollary 13-1-3. $\Delta APB \cong \Delta BPM$ by S.S.S. (or S.A.S. or Hypotenuse-Leg), so that $m \angle APC = m \angle BPC$. Also, $m \angle APD = m\angle BPD$ by supplements of congruent angles. Therefore $m \widehat{AC} = m \widehat{BC}$ and $m \widehat{AD} = m \widehat{BD}$, by the definition of measure of an arc. Hence $\overline{CD}$ bisects $\widehat{ACB}$ and $\widehat{ADB}$.

10. Case I: Draw the diameter from P. Since the diameter is perpendicular to the tangent it is perpendicular to $\overleftrightarrow{AB}$, by Theorem 9-11. By the theorem of Problem 9, $m\widehat{AP} = m \widehat{BP}$.

Case II: Draw the diameter perpendicular to the secants. By Case I, $m \widehat{AP} = m \widehat{BP}$ and $m \widehat{CP} = m\ CP = m \widehat{DP}$. By subtraction, $m \widehat{AC} = m \widehat{BD}$.

Case III: The diameter from P will have Q as its other endpoint, by Theorem 9-11 and Theorem 13-1. Then the two arcs are semi-circles having equal measures, by definition.

Alternate proofs involve drawing radii to form congruent triangles, or drawing chords which are transversals and using alternate interior angles.

11. By Theorem 13-6, $m \angle A = \frac{1}{2} m \overset{\frown}{BDC}$. Since $m \angle A = 90$, then $m \overset{\frown}{BDC} = 180$, and $\overset{\frown}{BDC}$ is a semi-circle. Hence, by definition, $\overset{\frown}{BAC}$ is a semi-circle.

12. By Problem 5 we know there is a circle through A,B,C. Suppose vertex D is in the exterior or the interior of this circle. Let $\overleftrightarrow{CD}$ intersect this circle in D'. Then ABCD' is inscribed in the circle, and, by Problem 6, $\angle BAD'$ is supplementary to $\angle C$. But $\angle BAD$ is supplementary to $\angle C$ by hypothesis. Therefore, $\angle BAD' \cong \angle BAD$, since supplements of the same angle are congruent. Hence $\overleftrightarrow{AD} = \overleftrightarrow{AD'}$ and D = D'.

13. Since $\overline{AC}$ and $\overline{BD}$ are tangent at the endpoints of a diameter, then $\overline{AC} \parallel \overline{BD}$. Also, AC and BD are segments of chords in the larger circle which are congruent by Theorem 13-2. By Corollary 13-1-3, the radii $\overline{OA}$ and $\overline{OB}$ bisect these chords, so that $\overline{AC} \cong \overline{BD}$. Therefore quadrilateral ADBC is a parallelogram, by Theorem 9-19. But the diagonals of a parallelogram bisect each other, so that $\overline{AB}$ and $\overline{CD}$ bisect each other at some point, P. Now O is the midpoint of $\overline{AB}$, so P = O, and C, O,D are collinear, making $\overline{CD}$ a diameter.

Other proofs are possible.

Problem Set 13-4b.

1. (See Teacher's Commentary.)

2. a. Since chords $\overline{AF}$ and $\overline{BH}$ are congruent, they cut off congruent minor arcs $\overset{\frown}{HAB}$ and $\overset{\frown}{FBA}$. By Theorem 13-5, $m\overset{\frown}{HA} + m\overset{\frown}{AB} = m\overset{\frown}{FB} + m\overset{\frown}{AB}$, and so $m\overset{\frown}{HA} = m\overset{\frown}{FB}$.

   Therefore,

   $AH = BF$ by Theorem 13-8.

   b. $m\angle A = m\angle B$ and $m\angle AHB = m\angle BFA$ by Corollary 13-6-2.

   Then $\triangle AMH \cong \triangle BMF$ by A.S.A.

3. Since ABCD is a square, $\overline{DA} \cong \overline{AB} \cong \overline{BC}$, and therefore, $\overset{\frown}{DA} \cong \overset{\frown}{AB} \cong \overset{\frown}{BC}$ by Theorem 13-7. Then $m\angle DEA = m\angle AEB = m\angle BEC$ since they are inscribed angles which intercept congruent arcs in the same circle.

4. a. $\angle BAC$.

   b. $\angle CAF$.

   c. $\angle ADB$, $\angle BAF$.

   d. $\angle DAF$.

   e. $\angle DCB$.

   f. $\angle ADC$.

   g. $\angle DCA$, DBA.

   h. $\angle DAF$.

   i. $\angle EAB$.

   j. $\angle DBC$.

5. Since $m\overset{\frown}{PB} = 120$, $m\angle BPC = 60$ by Theorem 13-9. $\overline{PQ} \perp \overline{CP}$, so that $m\angle BPQ = 30$. $\triangle APQ$ is a 30°-60° right triangle. Hence, $AP = 4\sqrt{3}$.

6. Draw the common tangent at H. Then the angle formed by the tangent at H and line u is measured by the same arc as the angles formed by the line u and the tangents at M and N. Then the tangents at M and N are parallel by corresponding angles in one case and by alternate interior angles in the other case.

7. Proof: Draw $\overline{PB}$. By Theorem 13-6, $m\angle BPR = \frac{1}{2} m\overset{\frown}{BR}$. By Theorem 13-9, $m\angle BPT = \frac{1}{2} m\overset{\frown}{PB}$. But $m\overset{\frown}{BR} = m\overset{\frown}{PB}$, so $m\angle BPR = m\angle BPT$. $\overline{BF} \perp \overleftrightarrow{PT}$ and $\overline{BE} \perp \overleftrightarrow{PR}$ by definition of distance from a point to a line. $PB = PB$, so $\Delta PBE \cong \Delta PBF$ by A.A.S. Therefore, $BE = BF$, which was to be proved.

8. Draw $\overline{BC}$, forming ΔBCE. Then, $m\angle DEB = m\angle C + m\angle B = \frac{1}{2} m\overset{\frown}{DS} + \frac{1}{2} m\overset{\frown}{AC} = \frac{1}{2}(m\overset{\frown}{DB} + m\overset{\frown}{AC})$.

9. Draw $\overline{BC}$, forming ΔBCE. Then
$m\angle E = m\angle ABC = m\angle C = \frac{1}{2} m\overset{\frown}{AC} - \frac{1}{2} m\overset{\frown}{BD} = \frac{1}{2}(m\overset{\frown}{AC} = m\overset{\frown}{BD})$.

10. The proof is the same as for Problem 9, except that Theorem 13-9 is used to get the measure of one angle in each case.

11. $m\overset{\frown}{BC} = 30$.
$m\overset{\frown}{CD} = 30$.
$m\angle K = 25$.
$m\angle E = 30$.
$m\angle BAD = 30$.
$m\angle AGE = 70$.
$m\angle DGE = 110$.
$m\angle ADK = 140$.

12. $m\overset{\frown}{DA} = 88$ and $m\overset{\frown}{BC} = 122$.
$m\angle EDC = m\angle DBC = 31$.
$m\angle CMD = m\angle AMB = m\angle ABC = 75$.
$m\angle DMA = m\angle CMB = 105$.
$m\angle FDB = m\angle DCB = 88$.
$m\angle ACB = m\angle ACD = m\angle DBA = 44$.
$m\angle CAB = m\angle CDB = 61$.
$m\angle DCE = 92$.
$m\angle DEC = 57$.
$m\angle DFA = 48$.
$m\angle CAF = 119$.
$m\angle CDF = 149$.

13. a. By Corollary 13-6-2, $m\angle ADP = m\angle ECP$ and $m\angle DAP = m\angle CBP$. Hence $\triangle APD \sim \triangle BPC$ by A.A.

b. Since similar triangles have corresponding sides proportional, $\frac{AP}{PB} = \frac{PD}{PB}$. Clearing of fractions we have

$AP \cdot PC = PB \cdot PD$.

14. a. By Theorem 13-9, $m\angle DAC = \frac{1}{2}\, m\overset{\frown}{AC}$, and by Theorem 13-6, $m\angle B = \frac{1}{2}\, m\overset{\frown}{AC}$. Therefore $m\angle DAC = m\angle B$. Since $\angle D$ is common to the triangles, $\triangle ABD \sim \angle CAD$ by A.A.

b. Since similar triangles have corresponding sides proportional, $\frac{BD}{AD} = \frac{AD}{CD}$. Clearing of fractions we have

$BD \cdot CD = (AD)^2$.

15. $m\angle a = \frac{1}{2}(m\overset{\frown}{AV} = m\overset{\frown}{DU}) = \frac{1}{2}(m\overset{\frown}{VB} - m\overset{\frown}{UC})$,

so $m\overset{\frown}{AV} + m\overset{\frown}{UC} = m\overset{\frown}{VB} + m\overset{\frown}{DU}$. Similarly working with

$\angle b$, $m\overset{\frown}{SD} + m\overset{\frown}{BT} = m\overset{\frown}{AS} + m\overset{\frown}{TC}$.

$$\begin{aligned}
\text{Now } m\angle PRQ &= \tfrac{1}{2}(m\overset{\frown}{UT} + m\overset{\frown}{SV}) = \tfrac{1}{2}(m\overset{\frown}{UC} + m\overset{\frown}{CT} + m\overset{\frown}{AS} + m\overset{\frown}{AV}) \\
&= \tfrac{1}{2}(m\overset{\frown}{UC} + m\overset{\frown}{AV}) + \tfrac{1}{2}(m\overset{\frown}{CT} + m\overset{\frown}{AS}) \\
&= \tfrac{1}{2}(m\overset{\frown}{VB} + m\overset{\frown}{DU}) + \tfrac{1}{2}(m\overset{\frown}{SD} + m\overset{\frown}{BT}) \\
&= \tfrac{1}{2}(m\overset{\frown}{VB} + m\overset{\frown}{BT}) + \tfrac{1}{2}(m\overset{\frown}{SD} + m\overset{\frown}{DU}). \\
&= \tfrac{1}{2}(m\overset{\frown}{VT} + m\overset{\frown}{SU}) = m\angle QRV
\end{aligned}$$

Therefore $\angle PRQ$ is a right angle, by definition.

Problem Set 13-5.

1. Proof:

Statements	Reasons
1. $\overleftrightarrow{AC}$, $\overleftrightarrow{CE}$ and $\overleftrightarrow{EH}$ are tangents at B, D, and F respectively.	1. Given.
2. $CB = CD$. $EF = ED$.	2. Theorem 13-10.
3. $CB + EF = CD + DE = CE$	3. Adding in step 2.

2. By Theorem 13-11, $x(x + 13) = 4 \cdot 12.$

$$x^2 + 13x = 48.$$

$$x = 3.$$

3. Let BK = a. Then by Theorem 13-12,

$$a(a + 5) = 36.$$

$$a^2 + 5a - 36 = 0.$$

$$(a + 9)\ (a - 4) = 0.$$

$$a = 4.$$

$$BK = 4.$$

4. By Theorem 13-13 we have $x(19 - x) = 6 \cdot 8.$

$$x^2 - 19x + 48 = 0.$$

$$(x = 3)\ (x - 16) = 0$$

$$x = 3,$$

$$w = 19 - x = 16.$$

5. Proof:

Statements	Reasons
1. $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are tangent at A and C, respectively.	1. Given
2. ΔAOB and ΔCOB are right triangles.	2. Corollary 13-1-1.
3. $m\angle ABO = m\angle CBO = 60.$	3. $m\angle ABC = 120$, and Theorem 13-10.
4. $AB = \frac{1}{2}\ OB; \quad CB = \frac{1}{2}\ OB.$	4. Theorem 11-10.
5. $AB + CB = OB.$	5. Adding in step 4.

6. Proof: Since tangents to a circle from an external point have equal lengths,

SN = SP,

NR = RM,

$CL = CP,$

$DL = DM.$

Adding and grouping,

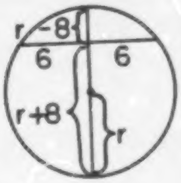
$(SN + NR) + (CL + DL) + (SP + CP) + (RM + DM)$, or

$SR + CD = SC + RD.$

7. Let r be the radius. Then, by Theorem 13-13,

$(r + 8)(r - 8) = 6 \cdot 6,$

$r^2 - 64 = 36, \quad r = 10.$



8. By Theorem 13-11, $7 \cdot AR = 6 \cdot 14,$

$AR = 6 \cdot 2 = 12.$

$AB = 12-7 = 5.$

9. Let the radius of the circle be r. Then by Theorem 13-12,

$4(2r + 4) = (12)^2.$

Hence $r = 16$.

10. Since all angles of the triangle have a measure of 60 the minor arc has a measure of 120, by Theorem 13-10. This leaves 240 for the measure of the major arc.

11. If n is the length of the shortest of the four segments, the rest of its chord would have to be the longest of the segments. Otherwise the product of the segments of this chord would certainly be less than the product of the segments of the other. Hence if it were possible to have consecutive integers for the lengths they would be labeled as shown. But in this case, by Theorem 13-13, it would be necessary that:

$n(n + 3) = (n + 1)(n + 2),$

or $n^2 + 3n = n^2 + 3n + 2$

or $0 = 2.$

Since this is impossible, the lengths of the segments cannot be consecutive integers.

12. Applying Theorem 13-12, we have $(AM)^2 = MR \cdot MS$ and $(MB)^2 = MR \cdot MS$. Hence $(AM)^2 = (MB)^2$ and $AM = MB$. Similarly $CN = ND$.

13. a. 4; 2 internal, 2 external.

    b. 1 internal, 2 external.

    c. 2 external only.

    d. 1 external only.

    e. None.

14. Draw radii $\overline{RA}$ and $\overline{QB}$. Let $\overline{AB}$ intersect $\overline{RQ}$ at P. $m\angle A = m\angle B = 90^\circ$, and $m\angle APR = m\ \angle BPQ$ by vertical angles. Therefore $\triangle APR \sim \triangle BPQ$ by A.A. This gives $\frac{RP}{QP} = \frac{RA}{QB}$. Now suppose $\overline{DC}$ meets $\overline{RQ}$ at point P'. Then, by a similar argument, we arrive at

    $\frac{RP'}{QP'} = \frac{RA}{QB}$ Hence $\frac{RP'}{QP'}$ $\frac{RP}{QP}$, and P and P' are both between R and Q. Therefore P' = P.

15. Problem 14 assures us that $\overline{AB}$ and $\overline{CD}$ meet $\overline{RQ}$ at the same point P. Therefore, by Theorem 13-10, $PA = PC$ and $PB = PD$. Adding, we have $PA + PB = PC + PD$, or $AB = CD$.

16. Draw $\overline{QR} \perp \overline{AP}$. In $\triangle PQR$, $RQ = \sqrt{(PQ)^2 - (PR)^2}$. Hence $RQ = 48$. But $AB = RQ$, since RQBA is a rectangle. Therefore $AB = 48$.

17. As in the previous problem, draw a perpendicular from the center of the smaller circle to a radius of the larger circle. By the Pythagorean Theorem, the distance between the centers is 39 inches.

18. Draw $\overline{QE} \perp \overleftrightarrow{PA}$. Since $PQ = 20$ and $PE = 7 + 9 = 16$, then $QE = 12 = AB$.

19. Let d be the required distance. By Theorem 13-12

$$d^2 = \frac{h}{5280}\ (8000 + \frac{h}{5280}).$$

$$d^2 = \frac{50}{33} h + \left(\frac{h}{5280}\right)^2 .$$

Now since h is very small compared to 5280, $\left(\frac{h}{5280}\right)^2$ is exceedingly small, and is not significant.

$$d = \sqrt{1.515\ h} = 1.23\ \sqrt{h}$$

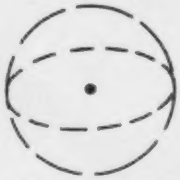
Hence, d is roughly $\frac{5}{4}\sqrt{h}$.

Solutions of Problems

Problem Set 14-1.

1. a. The locus is a circle with the given point as center and a two inch radius.

b. The locus is a sphere with the given point as center and a two inch radius.



2. a. Two lines parallel to the given line, at the given distance.

b. Four points.

c. Two points if the given distance is great enough, one if just half the distance between the points, and the empty set otherwise.

d. Two parallel segments linked at the ends by semicircles.

3. a. a sphere.

b. a cylindrical surface.

c. two planes.

d. four lines.

e. a circle if the distance is great enough. May be one point or the empty set.

f. a cylinder capped by 2 hemispheres.

4. a. a line (the ⊥ bisector).

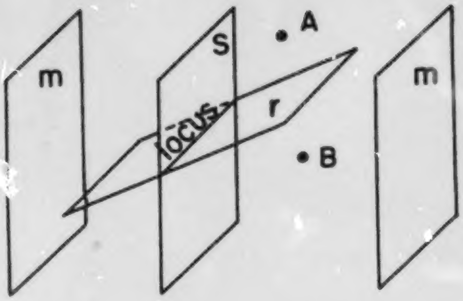
b. a line parallel to the two.

c. two lines, bisectors of the angles formed.

d. one point, the center of the circle through the points.

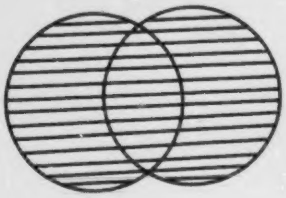
e. four points, centers of the four circles tangent to the lines.

5. a. a plane.
   b. a plane.
   c. a plane.
   d. two planes.
   e. a cone composed of lines through the foot of the perpendicular and making 45° angles with the given line.
6. a. 1. true.
      2. false.
   b. 1. true.
      2. false.
7. The pole should be placed at the point where the perpendicular bisectors of two sides of the triangle intersect.
8. The fountain should be placed at the intersection of the bisector of $\angle B$ and the perpendicular bisector of $\overline{DC}$.
9. The point is the intersection of $\overline{PQ}$ and the bisector of $\angle B$.
10. The vertices lie along the perpendicular bisector of $\overline{AB}$.
11. The point is the intersection of the perpendicular bisectors of two of the segments joining the pairs of points. If the points are collinears the two perpendicular bisectors will, of course, be parallel.
12. Points equidistant from two given points lie in a plane r_1. Points equidistant from two given parallel planes also lie in a plane r_2. In general, the intersection of two planes is a line, but if the two planes should be parallel, the intersection is the empty set or if the two planes should be equal the required locus is a plane. In summary the locus may be a line, a plane, or the empty set.

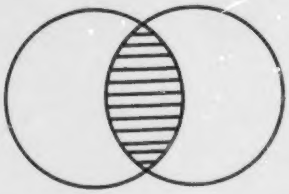


Given points A, B and parallel planes m and n.

13. a. The union of the interiors of two circles with 4 cm. radii and centers at the given points.



b. The intersection of the interiors of two circles with 4 cm. radii and centers at the given points.



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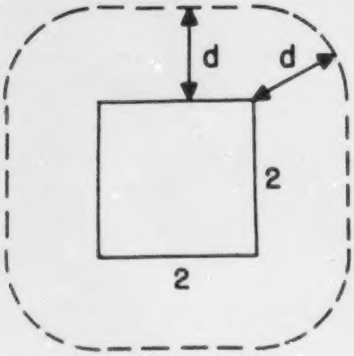
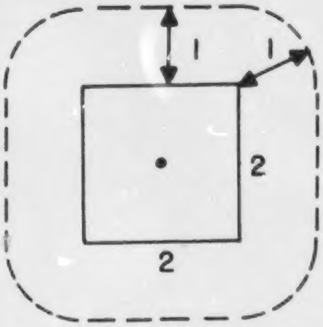
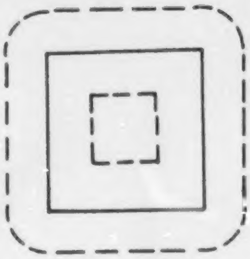


d >1

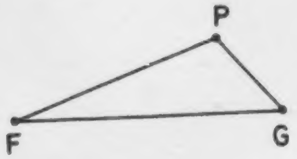


d= 1 (center of square is part of locus)

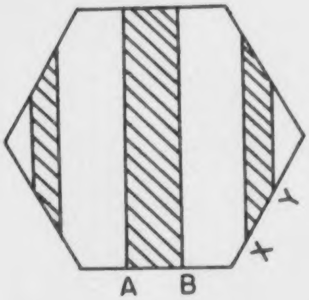


d < 1

16. Two pins are put in a drawing board, at F and G, and a loop of string placed around them and pulled taut by a pencil at P. As the pencil moves, always keeping the string taut, it describes a locus called an "ellipse". Describe the ellipse in geometric terms, just as the circle is described as the locus of points P such that CP = r, where C is a given point and r a given number.



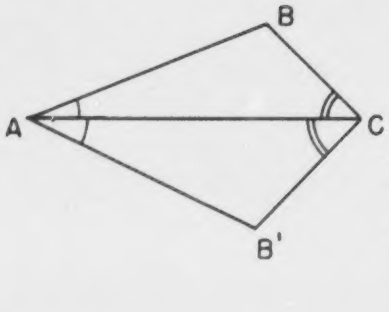
17. Tongue and grooves as in drawing with AB = xy.



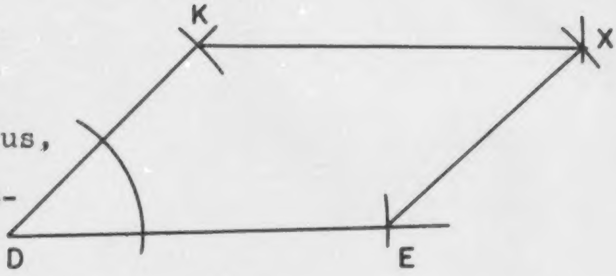
Problem Set 14-3.

1. Part d is not possible.

3.



5. Make $\overline{DE} \cong \overline{AB}$. Make $\angle D \cong \angle Q$ and $\overline{DK} \cong \overline{FH}$. Using E as center and the length FH as radius, strike an arc and with K as center and length AB as radius, strike another arc intersecting the first at X on the opposite side of $\overleftrightarrow{KE}$ from D. DEXK is the required parallelogram. (If both pairs of opposite sides of a quadrilateral are congruent, it is a parallelogram.)



6. a. If the length of the given segment $\overline{AB}$ is c, draw the circles center A and radius c, center B and radius c. Since $c + c > c$, these circles intersect at C, say, and the ΔABC is equilateral.

   b. If the radii of the two circles are changed to any length $r > \frac{c}{2}$, then $r + r > c$, and the ΔABC would be isosceles with base $\overline{AB}$.

7. Using the length OA as radius and O as center of a circle, construct an arc as shown above. Count down the number indicated (9 in this example) and draw another radius from O to the intersection of the arc and the nth line (9 in this case). The new radius, OB, congruent to the original segment, will be divided by the lines of the paper into congruent segments, which may be transferred to OA. We assume that the lines of the paper are parallel and thay they intercept congruent segments on one transversal (the margin of the sheet of paper). See Cor. 9-25-1.

9. Divide $\overline{AB}$ into three congruent segments. Construct an equilateral triangle with one of these segments as side.

10. Divide AB into five congruent segments. One of them can be used as the base.

11. No. We might have $a > b + c$, and it is necessary that the sum of any two of them be greater than the third.

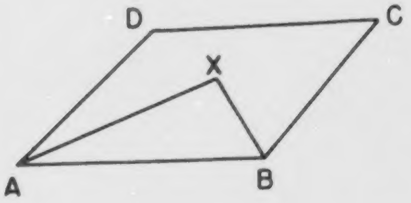
12. We must have $4 + 6 > x$; $4 + x > 6$; $6 + x > 4$. The last equation is true for all positive x, and the other two give $2 < x < 10$.

13. The required locus is the circle with $\overline{AB}$ as diameter, and with A and B omitted. If x is the indicated set, then A x B is a right triangle with $\overline{AB}$ as hypotenuse. If A X B is a right triangle, then the midpoint of the hypotenuse is equidistant from the vertices and so X is on the circle.

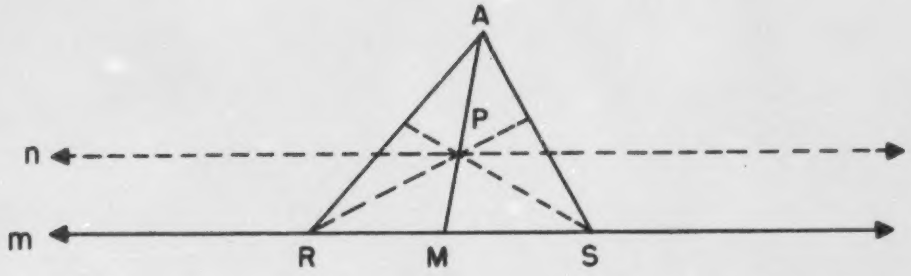
14. Midpoints of sides, midpoints of the segments of the altitudes (see 3 in directions) and feet of altitudes.

15. Since for each parallelogram $\angle A$ and $\angle B$ are supplementary, $m\angle \frac{1}{2} A + m\angle \frac{1}{2} B = 90$.

This means that the bisectors must be perpendicular to each other. Then the locus of X will be the circle whose diameter is $\overline{AB}$, except for points A and B.



16.

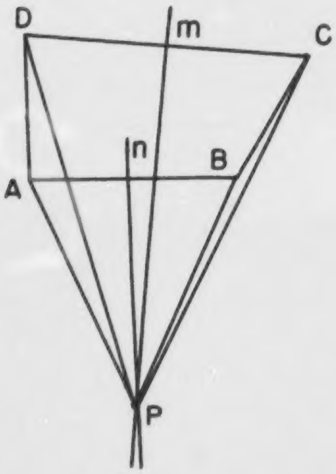


Since $\frac{AP}{AM} = \frac{2Z}{3}$, the locus of P is contained in a line n parallel to the given line m and such that

$$\frac{\text{Distance from A to n}}{\text{Distance from A to m}} = \frac{2}{3} .$$

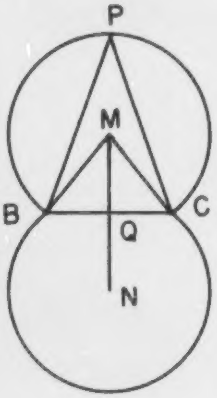
This follows from the converse of the Basic Similarity Also, if P is any point of n, we can choose any two points R and S on m such that RM = MS. Then P will be the centroid of $\triangle$ ARS, since $\overline{AM}$ is the median and $\frac{AP}{AM} = \frac{2}{3}$.

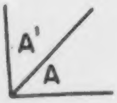
17. The correct drawing is as shown, with the ray $\overrightarrow{BA}$ not in the interior of $\angle$PBC. In this figure, m $\angle$CBA $\neq$ m $\angle$CBP - m $\angle$PBA.



18. ∠DAC is a right ∠ and so is ∠DBC, since they are both inscribed in a semi-circle. In Δ CDQ, $\overline{DA}$ and $\overline{CB}$ are altitudes to two of the sides. The altitude to the third side contains Q by definition and R by the theorem on the concurrence of the three altitudes of a triangle. $\overleftrightarrow{QR} \perp \overleftrightarrow{PO}$.

19. The locus consists of two circular arcs with centers M and N, one on each side of $\overleftrightarrow{BC}$, in each of which an angle of measure m A can be inscribed. To construct one of these arcs note that m ∠BMC = 2m ∠BPC - 2m ∠A, and so m ∠QMC - m ∠A, and ∠BCM is complementary to A. So construct A' complementary to A, and ∠BCM and ∠CBM congruent to ∠A'. This determines M, the center of one of the required circles. The other center is obtained by simply **taking QN** = QM. (Q is the midpoint of $\overline{BC}$).

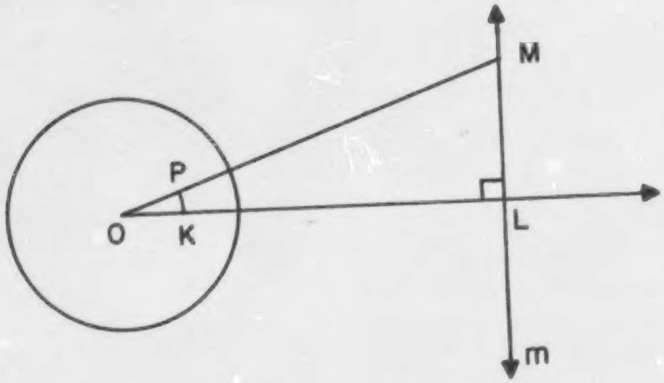




Strictly speaking, the locus is not exactly the two arcs, for the points B and C cannot be considered as points of the locus; it makes no sense to talk about ∠BPC if P = B or P = C.

If m ∠A > 90 the construction must be altered, but the principle is the same.

20.



Let $\overrightarrow{OL}$ be the ray $\perp$ m at L, and let K be determined on OL so that $OK \cdot OL = r^2$. If P is a point on the locus, then $OP \cdot OM = r^2 = OK \cdot OL$, so $\frac{OP}{OK} = \frac{OL}{OM}$. The triangles KOP and MOL have the angle O in common. Therefore they are similar, by the S.A.S. Similarity Theorem. Hence $\angle OPK$ is a right angle, and P lies on the circle with diameter $\overline{OK}$. Every point on this circle except O and K will yield a right triangle OPK similar to $\triangle OLM$, so $\frac{OP}{OK} = \frac{OL}{OM}$, that is, $OP \cdot OM = OK \cdot OL = r^2$.

Therefore the required locus is the circle with diameter $\overline{OK}$, except for the point O.

21. One simple way is to draw a part of a circle with center C, conveniently chosen so as to pass through A and cut $\overline{AB}$ at D. $\overrightarrow{DC}$ intersects the circle again at E, then $\overline{AE} \perp \overline{AB}$. There are many other ways.

22. Proof:

$\triangle AB'C \cong \triangle ABC'$ by S.A.S.

$\therefore$ Area $\triangle PBB'$; and Area $\triangle PCC'$; by subtraction.

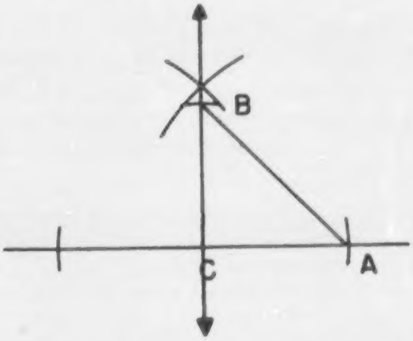
$BB' = CC'$; (Given.)

$\therefore PR = PK$ and so P is on the bisector of $\angle A$.

23. On an arbitrary ray through A lay off segments $\overline{AC}$ and $\overline{CD}$, with C between A and D, of lengths a and b. Through C draw $\overline{CX} \parallel \overline{DB}$. The $\triangle$s ACX and ADB are similar (A.A.) and have $\frac{AX}{XB} = \frac{a}{b}$.

24. Using the segment $\overline{AB}$ as a base construct the triangle (Theorem 14-6) whose sides are a,b,c. Construct the bisector of $\angle$ ACB, meeting opposite side in X.

Then $\frac{AX}{XB} = \frac{Aa}{b}$. Yes, by using a common multiplier for two of the segments and then following through. The division would still be in the same ratio.



Problem Set 14-4.

1. Construct $\overleftrightarrow{BC} \perp \overline{AC}$.

   Make $\overline{BC} \cong \overline{AC}$.

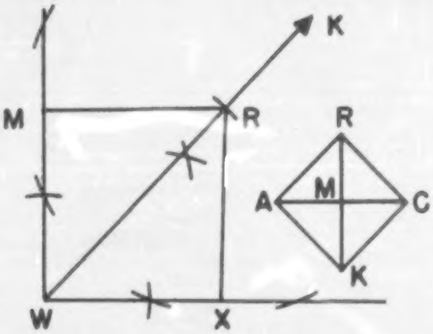
   $\triangle$ ABC is the required triangle.

2. In effect we have here "If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram" and we know that the opposite sides of a parallelogram are parallel.

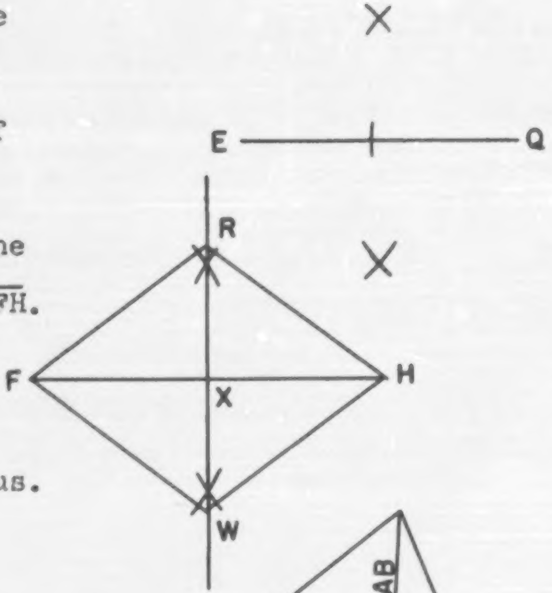
   Alternate: Use S.A.S. and alternate interior angles.

3. a. Bisect a 90° angle.
   b. Bisect a 60° angle (one angle of an equilateral triangle.)
   c. Bisect a 45° angle. (See a.)
   d. $90^\circ + 45^\circ$, or $180^\circ - 35^\circ$.
   e. $60^\circ + 60^\circ$ or $180^\circ - 60^\circ$.
   f. $30^\circ + 45^\circ$ or $90^\circ - 15^\circ$.
   g. $60^\circ + 45^\circ$, or $90^\circ + 15^\circ$.
   h. $22\frac{1}{2}^\circ$ is $\frac{1}{2}$ of 45°, and $67\frac{1}{2}^\circ = 45^\circ + 22\frac{1}{2}^\circ$.

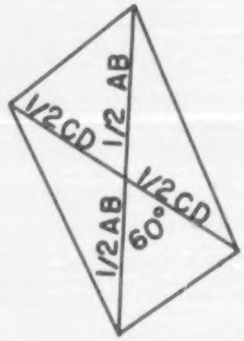
4. Make $\overline{WK}$ the bisector of a right angle, and $\overline{WR} \cong \overline{AC}$. From R construct perpendiculars to the sides of the right angle. WXRM is the required square. Alternate solution: Construct $\overline{RK} \perp$ bisector of $\overline{AC}$, and lay off $\overline{MR} \cong \overline{MK} \cong MA \cong MC$.



5. Make $\overline{FH} \cong \overline{AB}$. Construct the perpendicular bisector of $\overline{FH}$. Make $\overline{EQ} \cong \overline{CD}$. Bisect $\overline{EQ}$. Make XR = XW = $\frac{1}{2}$ EQ. FWHR is the required rhombus.

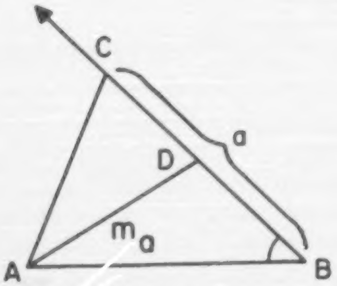


6.



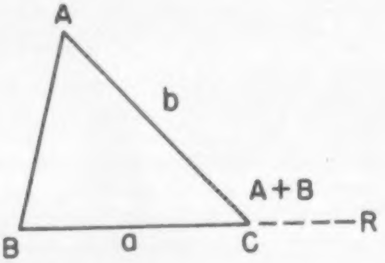
7. In the discussion that follows, each figure is just a sketch of the situation.

a. Construct $\angle B$ congruent to the given angle. Make BC = a. Find the midpoint D of $\overline{BC}$. Use D as center and m_a as radius to intersect $\overrightarrow{BA}$ at A.

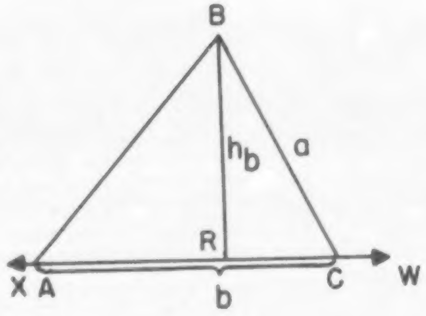


There are cases in which the construction is impossible and cases in which there are two solutions.

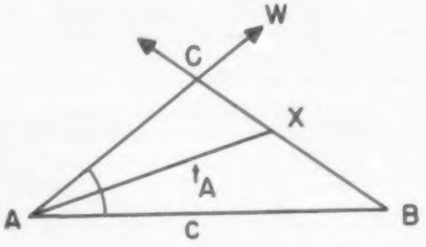
b. Construct $\angle ACR \cong \angle X$. Then $\angle ACB$ is the third angle of the triangle. Make CA = b and CB = a.



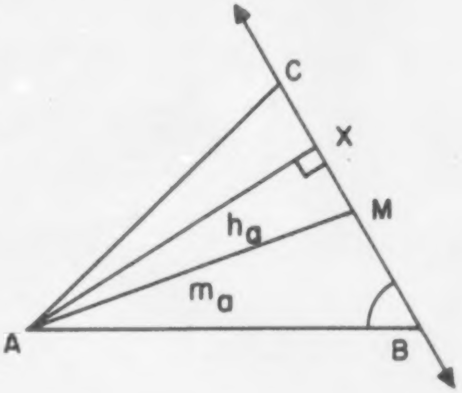
c. Construct a segment $\overline{RB}$ perpendicular to a "working line", $\overleftrightarrow{XW}$ at any convenient point and make RB = h_b. Using B as center and a as radius, construct an arc intersecting $\overleftrightarrow{XW}$ at C. Using C as center and b as radius construct an arc intersecting $\overleftrightarrow{XW}$ at A. (Two solutions in general, depending upon where A is taken, on the ray $\overrightarrow{CR}$ or on the opposite ray.)



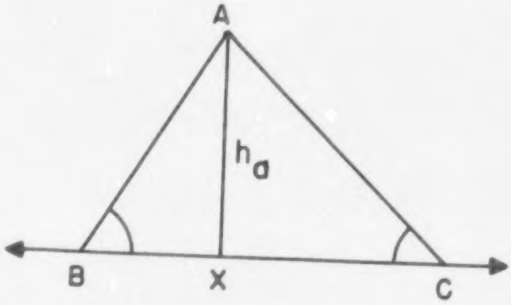
d. Construct $\angle WAB$ congruent to $\angle A$. Construct its bisector, $\overline{AX}$. Make AB = c. Connect B with X. The point at which $\overleftrightarrow{BX}$ meets $\overleftrightarrow{AW}$ is C. (There may be no solution, in case $\overleftrightarrow{BX} \parallel \overleftrightarrow{AW}$.)



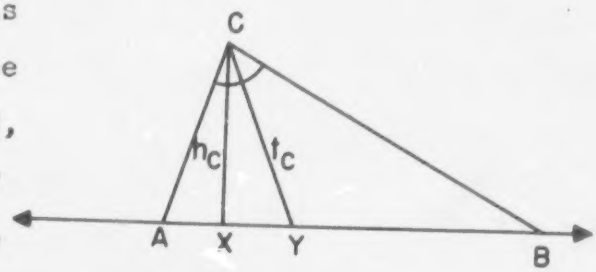
e. Since we are given $\angle B$ and since $M \angle AXB = 90$, we can construct $\angle XAB$. Then construct ΔABX by constructing $\overline{AX}$ (of length h_a) $\perp \overline{BC}$, and $\angle XAB$. Using A as center and m_a as radius, find M, then make MC = MB.



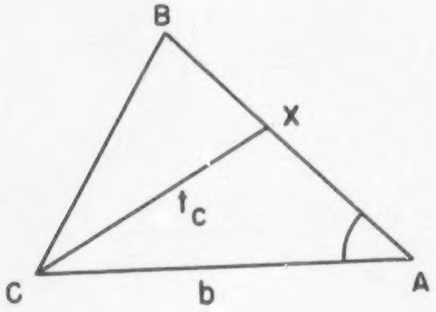
f. Start by constructing $\overline{AX}$ (of length h_a) perpendicular to a "working line", $\overleftrightarrow{BC}$. Since we are given $\angle B$ and since $m\angle BXA = 90$, $m\angle BAX$ can be easily constructed. Similarly $\angle CAX$ can be constructed. Construct these two angles at A.



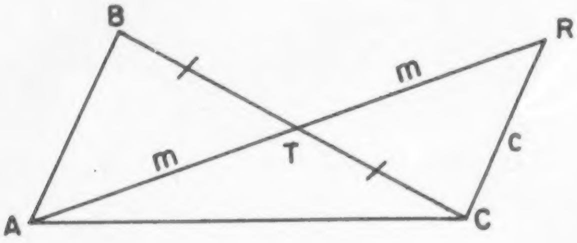
g. Construct $\overline{CX}$ (of length h_c) perpendicular to $\overleftrightarrow{AB}$. Use C as center and t_c as radius to cut $\overleftrightarrow{AB}$ at Y. Now since $\overline{CY}$ is the bisector of $\angle C$, construct on each side of $\overline{CY}$ an angle whose measure is $\frac{1}{2}$ $m\angle C$.



h. Construct an angle congruent to $\angle A$ and make AC = b. Using C as center and t_C as radius, find X. We now have $\angle$ XCA of measure $\frac{1}{2}$ m $\angle$ C. Construct $\angle$ XCB $\cong$ $\angle$ XCA.



8. Construct a triangle ARC with AC = b, CR = c, AR = 2m.



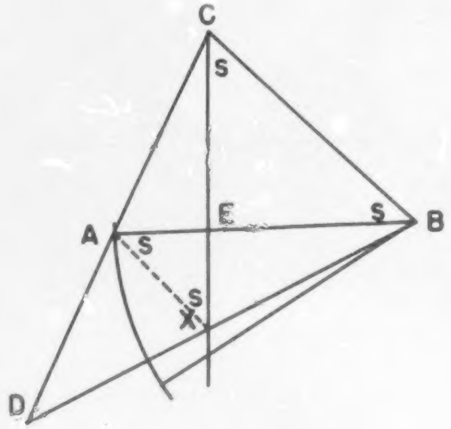
Bisect $\overline{AR}$ at T and extend $\overline{CT}$ through T to B so that CT = TB. Then Δ ABC is the required triangle. For ΔABT $\cong$ Δ RCT by S.A.S., so AB = CR = c. Clearly $\overline{AT}$ is the median and AT = m by construction.

9. We will take h, d, and e as the lengths of the given altitude and segments of the base. On $\overleftrightarrow{AF}$ as a "working line", make

XW = d and XQ = e. $\overline{QW}$ is the base of our triangle. Construct $\overrightarrow{XR} \perp \overleftrightarrow{AF}$ and on it make XP = h.

10. Construct AB = x and divide it into three congruent segments. At E (one of the trisection points) construct $\overleftrightarrow{CE} \perp \overline{AB}$. Make EX = AE, CE = BE. △DBC is the required triangle.

To prove that $\overline{BA}$ and $\overline{CX}$ are medians, draw $\overline{AX}$. Now,



△s EAX and ECB are isosceles triangles with congruent vertex angles and so angles s are all equal. Then $\overline{AX} \parallel \overline{CB}$ and

△EAX ~ △EBC with

$\frac{AX}{BC} = \frac{1}{2}$. Also △DAX ~ △DCB and $\frac{DA}{DC} = \frac{1}{2}$ so A is the midpoint. If $\overleftrightarrow{CA}$ and $\overleftrightarrow{BX}$ were parallel, AX would have to equal CB. This we have shown is not true, so $\overleftrightarrow{CA}$ and $\overleftrightarrow{BX}$ must intersect.

11. Bisect $\overline{KM}$ at L. Let $\overrightarrow{LN}$ be the other tangent to C from L. Bisect $\angle MLN$ with $\overrightarrow{LR}$. The intersection P of $\overrightarrow{LR}$ and the ray $\overrightarrow{MS} \perp m$ is the center of the required circle.

Proof: The common tangent $\overrightarrow{LN}$ meets m at L, and $LK = LN = LM$, so L is the midpoint of $\overline{KM}$. The right triangles PLN and PLM are congruent, so $\angle NLP \cong \angle MLP$ and $\overrightarrow{LP}$ bisects $\angle NLM$.

12. $m\angle ABD = 36$, so $m\angle BDA = 72$, and $\Delta BDA \sim \Delta CAB$.

Therefore, $\frac{AB}{DA} = \frac{1}{c}$,

But $m\angle DBC = 36$, so $DC = DB = AB = c$.

Hence, $c^2 = 1 - c$,

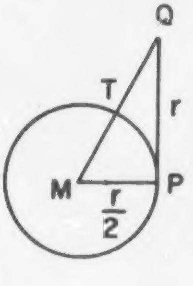
$$c^2 + c - 1 = 0$$

$$c = \frac{-1 \pm \sqrt{5}}{2}$$

Since c is positive, $c = \frac{\sqrt{5} - 1}{2}$.

13. The central $\angle$ subtended by one of the sides of the required pentagon has measure $360/5 = 72$. Bisect this angle to form ΔOAB, which will be a 36-72-72 Δ, with the two congruent sides of length r, the radius of the given circle. Our problem is therefore to find the length AB. By the preceding exercise, $AB = \frac{\sqrt{5} - 1}{2} r$.

To construct a segment of this length, draw a circle of radius, $\frac{r}{2}$. At the endpoint P of any radius, erect a $\perp$ segment $\overline{PQ}$ of length r. Let $\overline{MQ}$ intersect the circle at T. Then TQ has the required length.



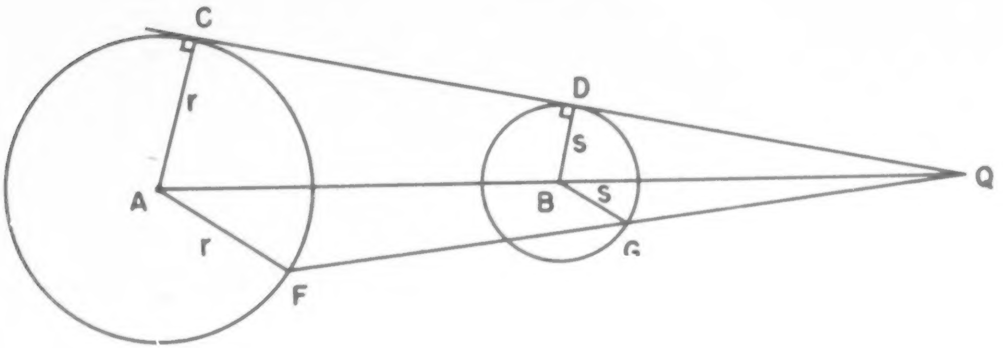
For in the right Δ MPQ, $MQ^2 = (\frac{r}{2})^2 + r^2 = \frac{5r^2}{4}$,

$$MQ = \sqrt{\frac{5}{2}} \quad r$$

$$\therefore TQ = MQ - MT = \frac{\sqrt{5}}{2} \ r - \frac{1}{2}\frac{r}{} = \frac{\sqrt{5} \ -1}{2} r.$$

To complete the construction, in the original circle, take any point A on the circle as center and $\overline{TQ}$ as radius and swing an arc, cutting the original circle at B. Do the same at B to get C. Then $\overline{AC}$ is one side of the required pentagon.

14.



The problem will be solved if we can find Q, the intersection of the common external tangent and the line determined by the centers. In the figure, $\triangle$s QDB and QCA are similar, being right triangles with a common acute angle.

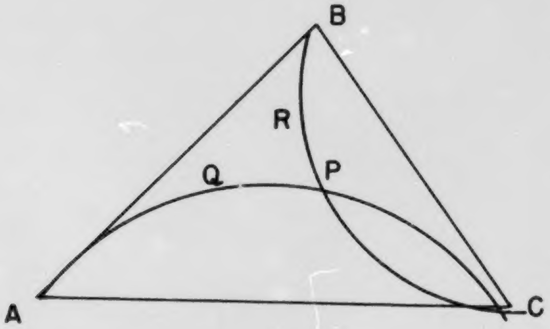
Therefore,

$$\frac{AQ}{BQ} = \frac{AC}{BD} = \frac{r}{s}.$$

Thus Q divides the segment $\overline{AB}$ externally in the ratio r/s. We can find Q by drawing a ray $\overrightarrow{AF}$ making a convenient angle with $\overrightarrow{AB}$, then drawing the ray $\overrightarrow{BG}$ parallel to $\overrightarrow{AF}$ and on the same side of $\overrightarrow{AB}$. Q is determined as the intersection of $\overrightarrow{FG}$ and $\overrightarrow{AB}$, since triangles AFQ and BGQ are similar, and

$$\frac{AQ}{BQ} = \frac{AF}{BG} = \frac{r}{s},$$ as desired.

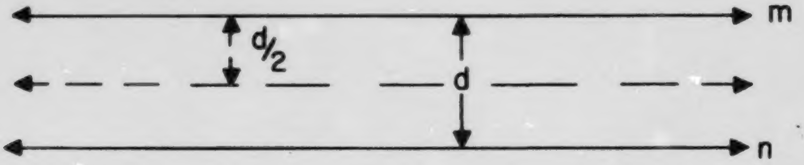
15.



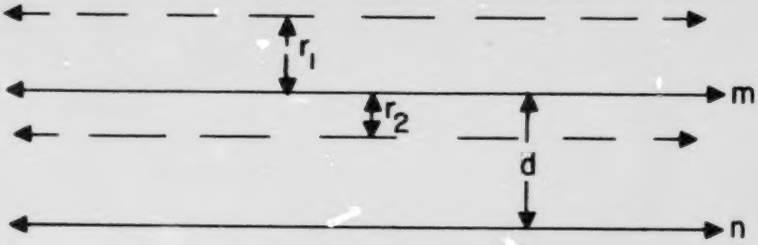
Let $\widehat{AQC}$ be an arc of 120°. Then m $\angle$ AQC = 120 for any position of Q on the arc. Similarly let $\widehat{BRC}$ be an arc of 120°. Hence if P is the point of intersection (other than C) of the two arcs, we have m$\angle$APC = M$\angle$BPC = 120. It follows automatically that m$\angle$APB = 120.

A complete analysis of this problem, including the case in which one angle has measure $\geqq 120$, is very complicated.

17. If $k = 1$, the locus of P is the line parallel to m and n and at a distance $\frac{d}{2}$ from each.



If $k < 1$, the locus of P is the union of two lines;



$$\frac{r_1}{r_1+d} = k, \qquad r_1 = \frac{k}{1-k}d.$$

$$\frac{r_2}{d-r_2} = k, \qquad r_2 = \frac{k}{1+k}d.$$

If $k > 1$, interchange the rôles of m and n.

18. Let $\overline{LK}$ be the segment through P perpendicular to m and n, and let $\overleftrightarrow{QR}$ be the ⊥ to m and n at R and S.

In the similar $\triangle$s PKD and PLB, $\frac{PK}{PL} = \frac{PD}{PB}$.

In the similar $\triangle$s PDA and PBC, $\frac{PD}{PB} = \frac{AD}{BC}$.

In the similar $\triangle$s QAD and QBD, $\frac{AD}{BC} = \frac{QA}{QB}$.

In the similar $\triangle$s QAR and QBD, $\frac{QA}{QB} = \frac{QR}{QS} = k$, a constant.

Therefore, $\frac{PK}{PL} = k$.

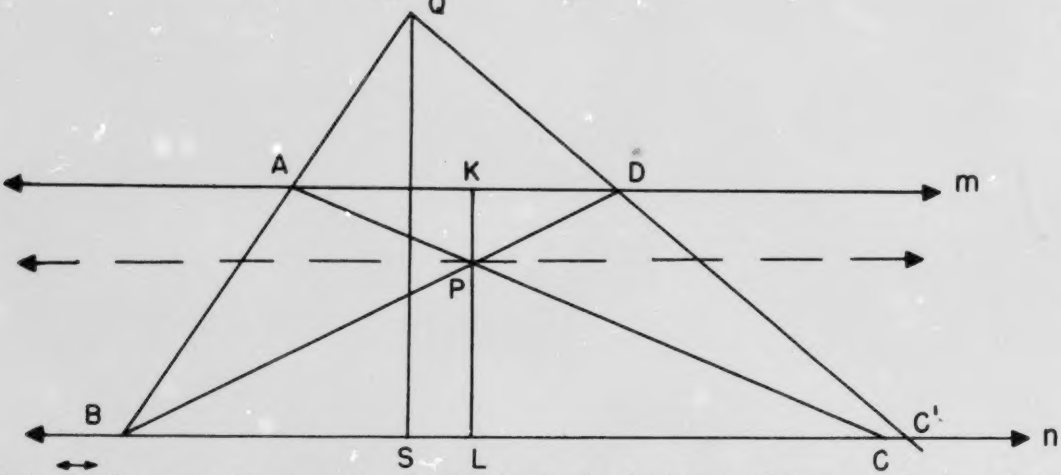
By the preceding exercise, the locus of P is contained in the union of two lines (or one, if $k = \frac{1}{2}$). Since P is between the lines m and n, the locus is contained in the line p parallel to m and at a distance $\frac{k}{1+k}$ d from m, where d is the distance between m and n.

To show that the locus is exactly the line p, let P be any point on p, so that

$$PK = \frac{k}{1+k}d .$$

Then $$PL = d - \frac{k}{1+k}d = \frac{1}{1+k}d, \text{ so}$$

$$\frac{PK}{PL} = k = \frac{QR}{QS} .$$



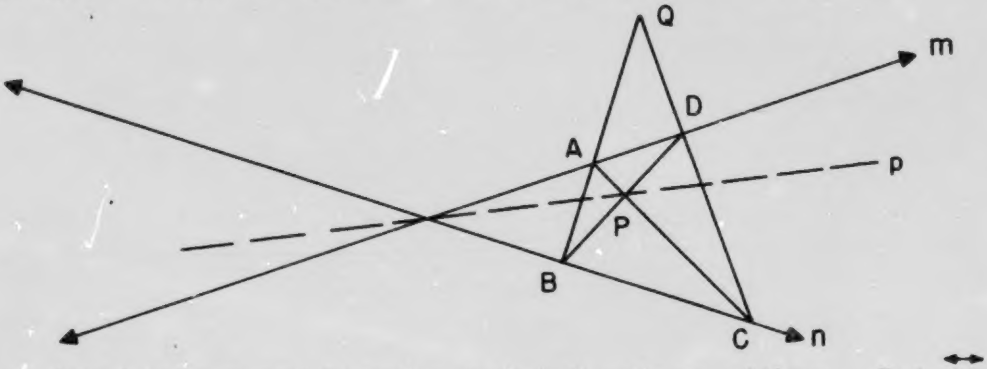
Let $\overleftrightarrow{QA}$ be any transversal through Q, intersecting m at A and n at B. Let $\overleftrightarrow{AP}$ intersect m at D and $\overleftrightarrow{BP}$ intersect n at C. Let $\overleftrightarrow{QD}$ intersect n at C'. We shall show that C' coincides with C.

$$\frac{PK}{PL} = \frac{PD}{PB} = \frac{AD}{BC} . \qquad \frac{QR}{QS} = \frac{QA}{QB} = \frac{AD}{BC'} .$$

Since $\frac{PK}{PL} = \frac{QR}{QS}$, $\frac{AD}{BC} = \frac{AD}{BC'}$, so C' = C.
(C' and C are on the same side of $\overleftrightarrow{QB}$).

This shows that every point P on p can be obtained by "crisscrossing" the points of intersection of two transversals, so the locus of "crisscrossed" points is exactly p.

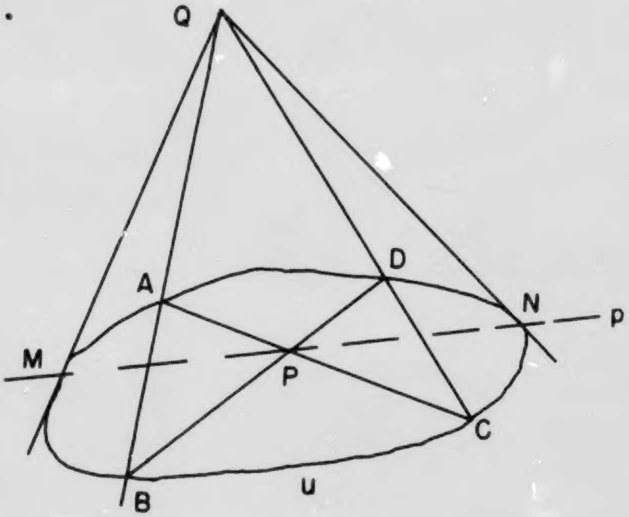
Note to the teacher: This problem is a special case of a theorem in projective geometry, known as the Theorem of Pappus:



Given two lines m and n and a point Q not on either. Let $\overleftrightarrow{QA}$ and $\overleftrightarrow{QD}$ be any lines through Q intersecting m and n in A and B, D and C, respectively, as in the figure. Then the locus of the point P obtained as the intersection of $\overleftrightarrow{BD}$ and $\overleftrightarrow{AC}$ is a line p.

This is, in turn, a special case of the following theorem:

Let u be any conic section and let Q be a point not on u. Let any two lines through Q intersect u in A and B, D and C, respectively, as in the figure.



Then the locus of the point P obtained as the intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ is a line p, called the polar of Q with respect to the conic u.

The points M and N where the polar intersects the conic are the points of tangency from Q. This gives us a neat way of finding the tangents to a conic (in particular, a circle) from an external point, using only a straight-edge. All we have to do is find two points P_1 **and** P_2 on the polar by "crisscrossing". We could even take the second point in the figure as the intersection of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$.

Problem Set 14-5.

1. $\triangle BPM \sim \triangle DPA$ with ratio of similarity $\frac{1}{2}$, since $\angle BPM \cong \angle DPA$; $\angle MBP \cong \angle ADP$, and $\frac{BM}{DA} = \frac{1}{2}$.

   Hence $\frac{BP}{DP} = \frac{1}{2}$ or $BP = \frac{1}{3}$ BD.

   A similar argument shows $DQ = \frac{1}{3}$ DB, so that P and Q are the trisection points of BD.

2. In the right triangle ABM, the ratio $\frac{AB}{BM} = 2$.

   If $m \angle BAM = 30$, this ratio would have to be $\sqrt{3}$, and hence, $m \angle BAM \neq \frac{1}{3} \cdot 90^\circ$.

   Hence the trisection of the segment $\overline{BD}$ would not lead to trisecting $\angle BAD$.

3. If you can get such a number you can also trisect an angle with straight-edge and compass.

4. Extend $\overline{QD}$ to meet $\overline{BA}$ at G, and drop $\overline{AH} \perp \overrightarrow{QP}$. Then

   $$\triangle QHA \cong \triangle QGA \cong \triangle QGB,$$

   from which the desired result follows. Notice that $\overrightarrow{QA}$ and $\overrightarrow{QG}$ are trisectors of $\angle PQR$.

Solutions of Problems

Problem Set 15-2.

1. Since the polygon is convex any diagonal (joining any vertex to any non-adjacent vertex) lies inside the figure. If we choose any vertex and draw all such diagonals from it, there will be (n-3) of these, making (n-2) triangles, all angles of which are parts of the angles of the polygon.

2. In each triangle, the sum of the measures of the angles is 180. For n triangles this sum will be n · 180. To get the sum of the measures of angles of the polygon it will be necessary to subtract 360, the sum of the measures of the angles whose vertices are at Q.

$$n \cdot 180 - 360 = n \cdot 180 - 2 \cdot 180 = (n-2)180.$$

3.	3.	120.	30.	60.
	4.	90.	45.	90.
	5.	72.	54.	108.
	6.	60.	60.	120.
	8.	45.	$67\frac{1}{2}$.	135.
	9.	40.	70.	140.
	10.	36.	72.	144.
	12.	30.	75.	150.
	15.	24.	78.	156.
	18.	20.	80.	160.
	20.	18.	81.	162.
	24.	15.	$82\frac{1}{2}$.	165.

4. a. Equilateral triangles; hexagons; three.

   b. Two pentagons and one decagon.

   c. Two 12-gons and one equilateral triangle.

5. Each exterior angle is 180 - (an interior angle.)

   Adding n of these we get

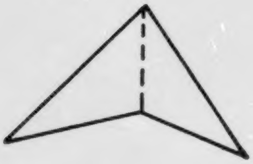
   $n \cdot 180$ - sum of interior angles

   or $n \cdot 180 - (n - 2)\,180 = 360$.

6. a. $\frac{n}{2} - 1$ or $\frac{n - 2}{2}$.

   b. $\left(\frac{n}{2} - 1\right) \cdot 360 = (n-2)180$.

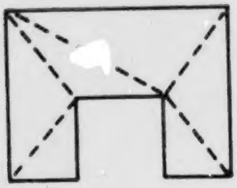
7. a.



$n = 4$,

$S = 2 \cdot 180 = (n-2)180$.

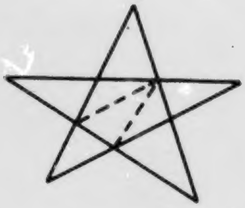
b.



$n = 8$,

$S = 6 \cdot 180 = (n-2)180$.

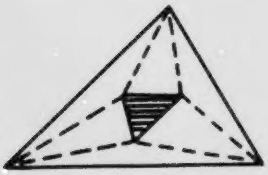
c.



$n - 10$,

$S = 8 \cdot 180 = (n-2)180$.

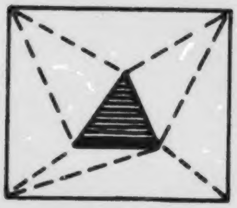
d.



$n = 6$,

$S = 6 \cdot 180 = n \cdot 180$.

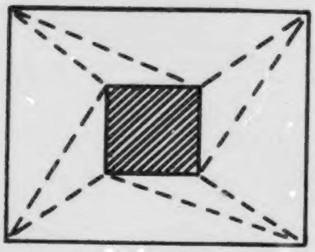
e.



$n = 7$,

$S = 7 \cdot 180 = n \cdot 180$.

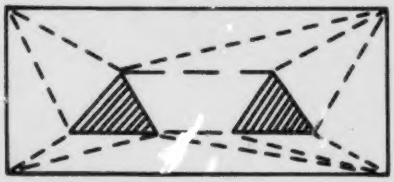
f.



$n = 8$,

$S = 8 \cdot 180 = n \cdot 180$.

g.



$n = 10$,

$S = 12 \cdot 180 = (n + 2)\ 180$.

In general, it seems that if a polygonal region has n sides, and if it has m "holes", then $S = (n - z + zm)\ 180$.

8. The angle sum is increased by 180 while the number of sides is increased by one.

9. Suppose one p - gon, one q - gon and one r - gon fit around a point. Then

$$\frac{(p-2)\ 180}{p} + \frac{(q-2)\ 180}{q} + \frac{(r-2)\ 180}{r} = 360.$$

This reduces to

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{2}$$

By trial we find the solutions

6	6	6
5	5	10
4	4	20
4	6	12
4	8	8
3	7	42
3	8	24
3	9	27
3	10	15
3	12	12

Problem Set 15-4.

1. a. $\frac{30\pi}{6} = 5\pi$. 5π inches.

   b. $\frac{30\pi}{4} = \frac{15\pi}{2}$ $\frac{15\pi}{2}$ inches.

   c. $\frac{30\pi}{5} = 6\pi$. 6π inches.

   d. $\frac{30\pi}{10} = 3\pi$. 3π inches.

2. Take 3.14 as the value of π. We get $C = 628 = 6.28R$. Hence $R = 100$ yards approximately.

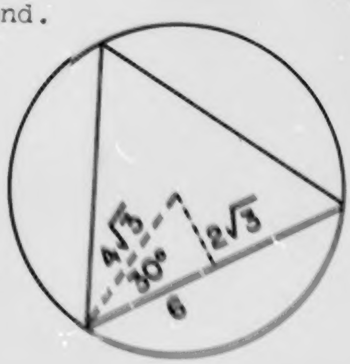
3. $16\pi - 4\pi = 12\pi$. The area of the washer is 12π sq. cm.

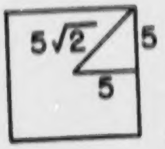
4. Area of first is 9 times area of second.

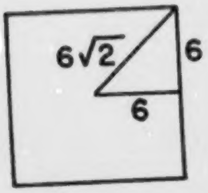
5. Radius = $4\sqrt{3}$ in.

   Circumference = $8\sqrt{3}\,\pi$ in.

   Area = 48π sq. in.







6. $\pi\ (5\sqrt{2}\ )^2\ - \pi\ (5)^2\ = 25\pi.$

Area is $25\ \pi$ sq. in.

7. a. $12\ \pi$ inches.

b. $12\sqrt{2}\ \pi$ inches.

8. $\frac{s}{2}^2$.

9. $AC^2 + BC^2 = AB^2.$

$\frac{\pi}{2}\ AC^2 + \frac{\pi}{2}\ BC^2 = \frac{\pi}{2}\ AB^2.$

$(r + g) + (h + s) = g + h + t.$

$r + s = t.$

10. $C = 2\ \pi\ r = 480{,}000\ \pi.$ The circumference is approximately 1,500,000 miles.

11. The formula gives $2\ \pi\ r = 6.28 \times 93 \cdot 10^6 = 584.10^6$ or 584 million miles, approximately.

Our speed is about 67,000 miles per hour.

12. a. Note that $r_1 = OA = OR = BP$ and $r_2 = OS = CP$. By successive use of the Pythagorean Theorem we get $r_1 = r\ \sqrt{2}$, $r_2 = r\ \sqrt{3}$, $r_3\ = r\sqrt{4}\ = 2r.$

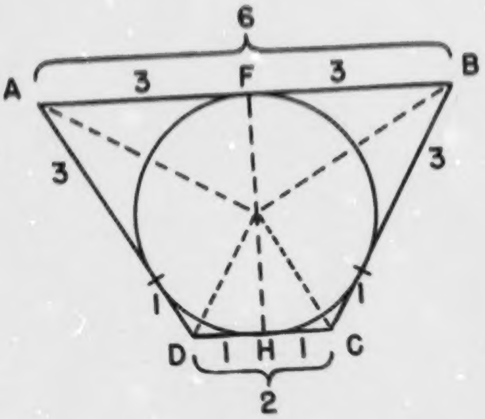
b. Now, using the area formula for a circle, we have

$a = \pi\ r^2;$

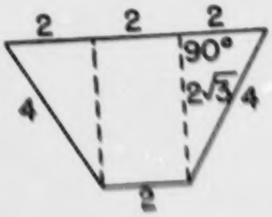
$b = \pi(r\sqrt{2}\ )^2\ - a = \pi r^2;$

$c = \pi(r\sqrt{3})^2 - (a+b) = 3\pi r^2 - 2\ \pi r^2 = \pi r^2;$

$d = \pi(2r)^2 - (a + b + c) = 4\ \pi\ r^2 - 3\ \pi r^2 = \pi r^2.$



13. From the second figure, $(4)^2 - (2)^2 = 12$, and so the altitude of the trapezoid is $2\sqrt{3}$. In the first figure, since the bases are parallel and tangent to the circle we see that $\overline{FH}$ (altitude of the trapezoid) must be a diameter, and so the radius is $\sqrt{3}$. Area of the circle is, then, 3π. Area of the trapezoid is $8\sqrt{3}$. The area outside the circle is $(8\sqrt{3} - 3\pi)$ eq. in. This is approximately 4 sq. in.



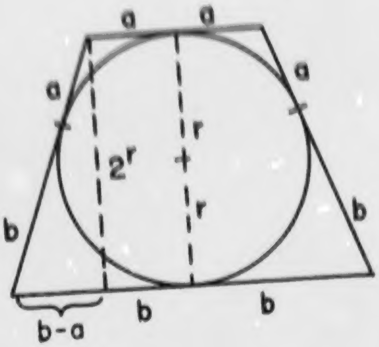
14. $(2r)^2 = (a + b)^2 - (b - a)^2$.

$4r^2 = a^2 + 2ab + b^2 - b^2 + 2ab - a^2$

$= 4\ ab$.

$r^2 = ab$.

$r = \sqrt{ab}$.



15. a. $\frac{1}{4}\ \pi \cdot 8^2 = 16\ \pi$.

b. $\frac{1}{9}\ \pi \cdot 9^2 = 9\ \pi$.

c. $\frac{1}{8}\ \pi \cdot 6^2 = \frac{9}{2}\ \pi$.

16. a. Area of sector $= \frac{1}{6}\ \pi \cdot 12^2 = 24\ \pi$.

Area of triangle $= \frac{12^2}{4}\sqrt{3} = 36\sqrt{3}$.

Area of segment $= 24\ \pi - 36\ \sqrt{3}$ or 13.03.

b. Area of sector $= \frac{1}{3} \pi \cdot 6^2 = 12 \pi$.

Area of triangle $= \frac{1}{2} \cdot 6\sqrt{3} \cdot 3 = 9\sqrt{3}$.

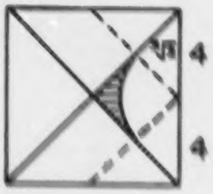
Area of segment $= 12\pi - 9\sqrt{3}$ or 36.71.

c. Area of sector $= \frac{1}{8} \pi \cdot 8^2 = 8 \pi$.

Area of triangle $= \frac{1}{2} \; 8 \cdot 4\sqrt{2} = 16\sqrt{2}$

Area of sector $= 8\pi - 16\sqrt{2}$ or 0.62.

17. To find one small shaded area subtract the area of a 90° sector whose radius is $2\sqrt{2}$ from the area of a square whose side is $2\sqrt{2}$.



$$(2\sqrt{2})^2 - \frac{\pi(2\sqrt{2})^2}{4} = 8 - 2\pi.$$

The area of the shaded area is $4(8 - 2\pi)$. This is approximately $7\frac{1}{2}$ sq. in.

18. $C = 2\pi r = 20.$ $P = 4s = 20$

$r = \frac{10}{\pi}.$ $s = 5.$

Area of circle $= \frac{100}{\pi}$. Area of square = 25.

approximately 32.

The area of the circle is greater by about 7 sq. in.

19. It is only necessary to find the square of the radius of the circle. If a radius is drawn to a vertex of the cross it is seen to be the hypotenuse of a right triangle of sides 2 and 6. The square of the radius is therefore $2^2 + 6^2 = 40$. The area of the circle is therefore 40π, 125.6 approximately.

The required area is therefore $125.6 - 80 = 40.6$.

20. Draw $\overline{PB}$ and $\overline{PC}$. The area of the annulus is $\pi(PC)^2 - \pi(PB)^2$, the difference of the areas of the two circles. This can also be written $\pi(PC^2 - PB^2)$. By Pythagorean Theorem, $PC^2 - PB^2 = BC^2$. Therefore the area of the annulus is $\pi\ BC^2$.

21. The section nearer the center of the sphere will be the larger.

$$r^2 = (10)^2 - (5)^2.$$

$$r_1^2 = (10)^2 - (3)^2.$$

$$\therefore r_1 > r.$$

22. Draw $\overline{BG} \perp \overline{AC}$. Then $GC = 6$, $AG = 24$. In the right triangle AGB, the length of the hypotenuse is twice the length of one leg, so $m\angle ABG = 30$, $m\angle BAG = 60$, and $CE = GB = 24\sqrt{3}$. The major arc $\widehat{CD}$ has the length $\frac{2}{3}(2\pi \cdot 30) = 40\pi$ and the minor arch $\widehat{EF}$ has the length $\frac{1}{3}(2\pi \cdot 6) = 4\pi$. Thus the total length of the belt is

$$2(24\sqrt{3}) + 40\pi + 4\pi = 48\sqrt{3} + 44\pi.$$

The belt is approximately 221 inches long.

23. The line must intersect opposite sides, otherwise one area will be too small.

$$\text{Area } ARSD = \frac{1}{2} h\ (a + c)$$

$$\text{Area } CSRB = \frac{1}{2} h\ (b + d).$$

$$a + c = b + d.$$

But $a + b = c + d$, so by subtraction,

$$c - b = b - c,$$

$$c = b.$$

If $\overline{AC}$ intersects $\overline{RS}$ in M $\triangle ARM \cong \triangle CSM$ by A.S.A., so AM = CM. Therefore $\overleftrightarrow{RS}$ contains M, the midpoint of the diagonal, which is the center of the rectangle.

Problem Set 15-7.

1. $5 \times 4 \times 3/4 = 15$. 15 cu. ft. of water in the tank.

   $\frac{15 \times 1728}{231} = 112$ approx. 112 gal. approximately.

2. $V = Ah = (\frac{8^2}{4}\sqrt{3})\ 10 = 160\sqrt{3}$.

   The volume is $160\sqrt{3}$ cu. in.

3. Since each lateral face is a rectangle, its area is the product of base and altitude. If e is the length of a lateral edge and S_1, S_2, S_3, ... are lengths of the sides of the prism base, then $A_1 = S_1 e$, $A_2 = S_2 e$, $A_3 = S_3 e$, ... Adding these areas to get the lateral area, $A = S_1 e + S_2 e + \ldots = (S_1 + S_2 + S_3 + \ldots)$ e. But $S_1 + S_2 + S_3 + \ldots = p$, the perimeter of the base. Therefore, $A = p\,e$.

4. Let the required perimeter be y. Since $52 = 10 \times y$, we have $y = 5.2$. The perimeter of the base is 5.2 in.

5. $QA = QB$, $m\angle VQA = m\angle VQB = 90$;

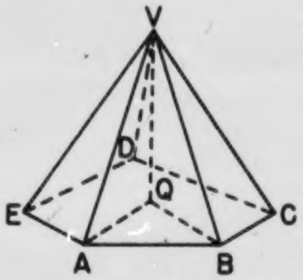
   $\triangle VQA \cong \triangle VQB$ by S.A.S.

   Hence VA = VB. Similarly

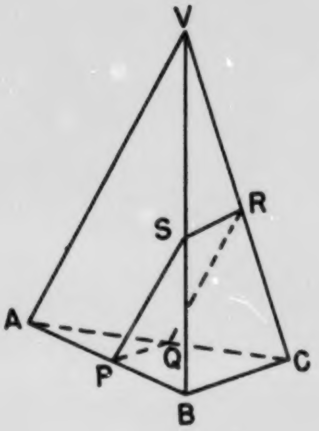
   VB = VC = VD ... AB = BC = CD

   ... by definition , so $\triangle AVB$

   $\cong \triangle DVC \cong \triangle CVD \cong \ldots$ by S.S.S.



6. Let P, Q, R and S be the midpoints of $\overline{AB}$, $\overline{AC}$, $\overline{VB}$ and $\overline{VC}$ respectively. Then $\overline{SR}$ and $\overline{PQ}$ are each parallel to $\overline{BC}$ and equal in length to $\frac{1}{2}$ BC. Therefore $\overline{SR}$ and $\overline{PQ}$ are parallel and equal in length making PQRS a parallelogram.



7. Let each edge of the base have length S. Each face is a triangle with base length s and altitude of length a.

Hence $A = \frac{1}{2}\,sa + \frac{1}{2}\,sa + \ldots\ldots$ or

$$A = \frac{1}{2}\,a\,(s + s + \ldots) = \frac{1}{2}\,a\,p.$$

8. The base can be divided into six equilateral triangles with side 12. Therefore, altitude $\overline{QF}$ of ΔABQ has length $6\sqrt{3}$. Since $QC = 9$, by Pythagorean Theorem, $CF = \sqrt{189}$. Hence the lateral area is $\frac{1}{2} \cdot 72 \cdot \sqrt{189} = 36\sqrt{189}$.
Now, $V = \frac{1}{3}\,Ah$, or $V = \frac{1}{3}\,(6\ \frac{s^2}{4}\sqrt{3})\,9$. Since $s = 12$, $V = 648\sqrt{3}$.

9. $1336 = \frac{1}{3} \cdot (18)^2 \cdot h$. or $h = 17$. The height is 17 feet.

10. Let $PK = a$ and $PB = b$. Draw altitude $\overline{PS}$. $\overline{PS} \perp JKLMN$ at R. $\overline{PB}$ and $\overline{PS}$ determine a plane which intersects JKLMN and ABCDE in $\overleftrightarrow{KR}$ and $\overleftrightarrow{BS}$ respectively. Since $JKLMN \parallel ABCDE$, $\overleftrightarrow{KR} \parallel \overleftrightarrow{BS}$. In ΔPBS, by the Basic Similarity Theorem, $\frac{PK}{PB} = \frac{PR}{PS}$. By Theorem 15-5,

$$\frac{\text{Area } JKLMN}{\text{Area } ABCDE} = \left(\frac{PR}{PS}\right)^2.$$ Hence, $$\frac{\text{Area } JKLMN}{\text{Area } ABCDE} = \left(\frac{PK}{PB}\right)^2 = \left(\frac{a}{b}\right)^2.$$

11. The lateral edges will also be bisected and therefore corresponding sides of the section and base will be in the ratio $\frac{1}{2}$, and (from prob. 10) the areas of the section and base in the ratio $\frac{1}{4}$. The volume of the pyramid above the section will be $\frac{1}{8}$ of that of the entire pyramid because its base has $\frac{1}{4}$ the area of that of the pyramid and its height is half as great. The solid below the plane will then have $\frac{7}{8}$ the volume of the entire pyramid and the ratio of the two volumes is $\frac{1}{7}$.

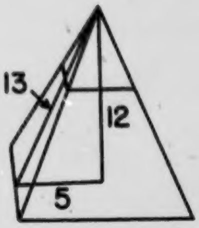
12. By Theorem 15-5

$$\frac{336}{x} = \frac{49}{16},$$

$$x = \frac{16 \cdot 336}{49} = \frac{768}{7} = 109\frac{5}{7},$$

Area FGHJK = $109\frac{5}{7}$ square inches.

13. The altitude of each face is 13 inches by the theorem of Pythagoras. Hence the lateral surface is $4(\frac{1}{2} \cdot 10 \cdot 13) = 260$ square inches.



The ratio of the area of the cross section 3 inches from the base is to the area of the base as $(\frac{9}{12})^2 = \frac{9}{16}$. Hence its area is $\frac{9}{16} \cdot 100 = 56.2$ square inches approximately.

14. The volume of the complete pyramid which is 60 feet tall is 320 cubic feet. The base of the smaller pyramid is 30 feet above the ground so the part of the 60 foot pyramid to be included contains $\frac{7}{8} \cdot 320$ of 280 cubic feet (see problem 11).

The small pyramid capping the monument has volume $\frac{1}{3} \cdot 4 \cdot 2$ or about 2.7 cubic feet. Hence the volume of the obelisk in cubic feet is 282.7.

Problem Set 15-9.

1. $V = \frac{1}{3}(9 \; \pi) \, 4 = 12 \, \pi$.

2. The number of gallons is $\frac{\pi \cdot 14^2 \cdot 30}{3 \cdot 31} = \frac{22 \cdot 14 \cdot 14 \cdot 30}{7 \cdot 3 \cdot 3 \cdot 7 \cdot 11} = \frac{80}{3} \;\; 26 = \frac{2}{3}$.

   (The factors of 231 are $3 \cdot 7 \cdot 11$. By using $\frac{22}{7}$ the computation can be simplified by reducing fractions.)

3. Let the radius of the base of each cylinder be r and the altitude be h. Then the volume of the cone in Figure 1 is $\frac{\pi r^2 h}{3}$. The volume of the two cones in Figure 2 is

   $$2 \left(\frac{\pi r^2}{3} \cdot \frac{h}{2} \right) = \frac{\pi r^2 h}{3}.$$

   The volumes are the same.

4. Subtract the volume of the inner cylinder from that of the outer. This gives

   $$16 \, \pi \, (2.8)^2 - 16 \, \pi \, (2.5)^2$$

   or

   $$16 \, \pi \, (2.8^2 - 2.5^2) = 16 \, \pi \, (2.8 - 2.5)(2.8 + 2.5)$$

   $$= 16 \, \pi (.3)(5.3) = 80.3 \text{ approximately.}$$

   Approximately 80.3 cu. in. of clay will be needed.

5. Let r be the radius of the base of the first can and h be its height. Then the radius of the second can is 2 r and its height is $\frac{h}{2}$. Then

   Volume of first can $= \pi \, r^2 \, h$.

   Volume of second can $= \pi \, (2r)^2 \, \frac{h}{2} = 2 \, \pi \, r^2 h$.

Since the volume of the second can is twice that of the first, and the cost is twice the cost of the first, neither is the better buy.

6. The volume of the pyramid is

$$\frac{20^2 \cdot 36}{3} = 4{,}800$$

The radius of the base of the cone is half the diagonal of the square, or $10\sqrt{2}$.
The area of the base of the cone is $\pi \cdot (10\sqrt{2})^2$ and the volume of the cone is $\frac{200\pi \cdot 36}{3} = 7{,}536$ approximately.

7. Since $r = 1$, the volume of the ice cream is $\frac{4}{3}\pi$ and the volume of the cone is $\frac{5}{3}\pi$. Therefore the cone will not overflow.

8. $\pi r^2 h - \frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^2 h$.

9. a. The volume of a cube of edge s is s^3; the volume of a cube of edge $(4s)^3$ or $64s^3$. Hence the ratio of the volumes is 64 to 1.

   b. If R and $4R$ are radii of the moon and the earth the volumes have the ratio $\frac{\frac{4}{3}\pi R^3}{\frac{4}{3}\pi (4R)} 3 = \frac{1}{64}$.

10. The altitude of the cone is r plus the hypotenuse of a 30°, 60°, 90° triangle of short side r. So the altitude is $3r$. Since the figure can be divided into 30° - 60° congruent right triangles,
The radius of base of the cone is $\sqrt{2r^{2} - r^2} = r\sqrt{3}$, so the area of its base is $\pi \cdot 3r^2$.
The volume of the cone is therefore

$$\frac{1}{3} \cdot 3\pi r2 \cdot 3r = 3\pi r^3.$$

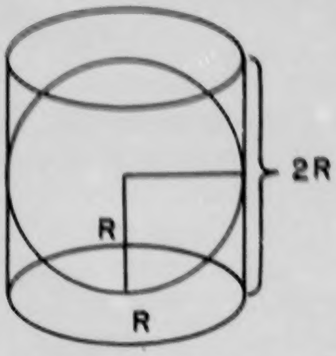
11. The ratio of the volumes is the cube of the ratio of the altitudes, so

$$\frac{V_2}{V_1} = \frac{2}{5}^3 = \frac{8}{125} = .064$$

Hence $V_2 = .064 \times 27 = 1.73$ approx.

12. The area of a hemisphere is one-half the area of a sphere, or $2\pi r^2$. Since the area of the floor is πr^2, twice as much paint is needed for the hemisphere, or 34 gallons.

13.



Volume of cylinder is $\pi R^2 \cdot 2R = 2\pi R^3$. $\frac{2}{3} \cdot 2\pi R^3 = \frac{4}{3}\pi r^3$, which is the formula for the volume of the sphere by Theorem 15-14.

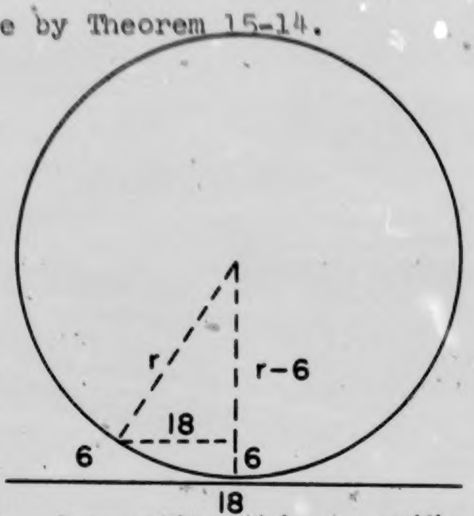
14. Let r be the radius of the tank in feet.

$r^2 = 18^2 + (r - 6)^2$.

$r^2 = 324 + r^2 - 12r + 36$.

$12r = 360$.

$r = 30$. The radius is 30 feet.



Using $V = \frac{4}{3}\pi r^3$ the volume of the tank in cubic feet is $\frac{4\pi \cdot 30^3}{3}$. Converting this to cubic inches, finding the number of gallons contained, and dividing by 10,000, the number of hours a tank full will last is

$$\frac{4 \cdot \pi \cdot 27000 \cdot 1728}{3 \cdot 231 \cdot 10000}$$, or about 85 hours.

15. Let V be the original volume and R the original radius, the new volume and r the new radius.

Then
$$\frac{V}{u} = \frac{2}{1} = \frac{\frac{4}{3}\pi R^3}{\frac{4}{3}\pi r^3} = \frac{R^3}{r^3}$$

Therefore $\frac{R^3}{r^3} = \frac{2}{1}$ or $\frac{R}{r} = \frac{\sqrt[3]{2}}{1}$

Hence $r = \frac{R}{\sqrt[3]{2}} = \frac{\sqrt[3]{4}\,R}{2}$.

Since $\sqrt[3]{4}$ is approximately 1.6

r is approximately $\frac{4}{5}$ R.

16. The volume of the frustum is the difference of the volumes of two pyramids. Hence their heights must be found. If x represents the height of the upper pyramid

$\frac{x}{4} = \frac{x + 8}{6}$, from which

$x = 16$ and $x + 8 = 24$.

$$\frac{1}{3}\pi\, 6^2 \cdot 24 - \frac{1}{3}\pi \cdot 4^2 \cdot 16 = \frac{608\,\pi}{3}\ .$$

The volume is approximately 636 cubic inches.

Solutions of Problems

Problem Set 16-3.

1. "Cartesian" is suggested by the discoverer, Descartes.
2. (0,0).
3. -3.
4. The origin, or (0,0).
5. (2,1) and (2,0).
6. a. IV. b. II, c. I. d. III.
7. One of the coordinates must be 0.
8. D, B, C, A.
9. C, A, D, B.
10. a. II. b. I. c. IV. d. III.
    e. IV. f. I. g. II. h. III.
11. a. y-axis, x-axis, z-axis.
    b. xz-plane, yz-plane, xy-plane.
    c. 4, 2, 3.

Problem Set 16-4.

1. a. 7. b. -1. c. y_1.
2. a. 6. b. -3. c. x_1.
3. West.
4. a. 2 b. 2. c. 3.
    d. They have the same y-coordinate.
    e. If two points in a plane have the same y-coordinate, the distance between them is the absolute value of the difference of their x-coordinates.
    f. No.

5. a. 3. b. 2. c. 4.

d. $|y_1 - y_2|$ or $|y_2 - y_1|$.

e. They have the same x-coordinate.

f. If two points in a plane have the same x-coordinate, the distance between them is the absolute value of the difference of their y-coordinates.

6. (2,3); (-1, -5); (3, -1).

7. PA = 2, QA = 2.

PB = 5, QB = 3.

PC = 7, QC = 3.

8. -1, 3/5, 3/7.

9. 1/15.

10. a. 1/3. b. -3. c. 7/4. d. 3/4. e. -15/8.

f. 8/15. g. -1. h. -3.

11. a. 6. b. -2.5.

12. a. Yes b. No.

13. a. -1. b. -3/2. c. $\frac{a - b}{2b}$.

14. Slope of $\overleftrightarrow{AB}$ is 96/96 = 1. Slope of $\overleftrightarrow{BC}$ is 100/100 = 1. Therefore $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ coincide.

15. Slope of $\overleftrightarrow{AB}$ is 96/96 = 1; slope of $\overleftrightarrow{CD}$ is 1/1 = 1. We are tempted to say that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, but we must make sure that they are actually two different lines. We test by finding slope of $\overleftrightarrow{AC}$, which is 101/101 = 1. Hence, $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ must coincide so that C is on $\overleftrightarrow{AB}$ and the lines can't be parallel. It easily follows that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ coincide.

16. Draw the segment which passes through (4, 3) and the origin.

Problem Set 16-5.

1. $\overline{AB} \parallel \overline{CD}$, $\overline{BC} \parallel \overline{AD}$.

2. Slope of $\overline{AB} = -\frac{2}{3}$, slope of $\overline{CD} = -\frac{2}{3}$.

   Slope of $\overline{BC} = -3$, slope of $\overline{DA} = -3$.

   Therefore opposite sides are parallel and the quadrilateral is a parallelogram.

3. $L_1 \perp L_3$ and $L_2 \perp L_4$, by Theorem 16-3.

4. The second is a parallelogram, as can be shown from the slopes of $\overline{PQ}$, $\overline{RS}$, $\overline{QR}$, and $\overline{PS}$, which are respectively 2/3, 2/3, -1/5, -1/5. The first is not a parallelogram since the slopes of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{AD}$ are respectively 4, 1/2, 5, and 3/8.

5. a. Slope of $\overline{AB} = -2/7$.

   Slope of $\overline{BC} = 2/9$

   Slope of $\overline{AC} = 0$.

   b. Slope of altitude to $\overline{AB} = 7/2$.

   Slope of altitude to $\overline{BC} = -9/2$.

   The altitude to $\overline{AC}$ has no slope; it is a vertical segment.

6. Both $\overline{AB}$ and $\overline{CD}$ have the same slope, -1, and are therefore parallel. $\overline{AD}$ and $\overline{BC}$ have different slopes. Therefore the figure is a trapezoid. Diagonal $\overline{AC}$ is horizontal since its slope is 0. Diagonal $\overline{BD}$ is vertical. A vertical and a horizontal line are perpendicular.

7. The slope in both cases is the same, -1/3, so that the lines are parallel.

8. The slope of the first line is b/a. The slope of the second is -a/b. Since the negative reciprocal of b/a is - a/b, the lines are perpendicular.

9. Application of the slope formula shows that the slope of $\overline{XY}$ is b/a, and that of $\overline{XZ}$ is -a/b. By Theorem 16-3 $\overline{XY} \perp \overline{XZ}$. Hence angle X is a right angle.

10. $\angle$ PQR will be a right angle if $\overline{PQ} \perp \overline{QR}$.

$\overline{PQ}$ will be perpendicular to $\overline{QR}$ if their slopes are negative reciprocals; that is, if:

$$\frac{-6 \quad -2}{5 - 1} = -\frac{b - 5}{b + 6}$$

from which b = -17.

Problem Set 16-6.

1. a and b. AB = 1, AC = 3, AD = 4.5, BC = 4, BD = 3.5, CD = 7.5.

2. a. $|x_2 - x_1|$ or $\sqrt{(x_2 - x_1)^2}$.

   b. $|y_2 - y_1|$ or $\sqrt{(y_2 - y_1)^2}$.

3. a. 5. b. 5. c. 13. d. 25.

   e. 17. f. $\sqrt{2}$ g. 89. h. $5\sqrt{5}$.

4. a. $(y_2 - y_1)^2 + (x_1 - x_2)^2$ b. $x^2 + y^2 = 25$.

5. By the distance formula RS = 5, RT = $\sqrt{2}$ and ST = 5. Since ST = RS the triangle is isosceles.

6. $\triangle$ DEF will be a right triangle with $\angle$ D a right angle only if $DE^2 + DF^2 = EF^2$. This is the case since $DE^2 = 5$, $DF^2 = 45$ and $EF^2 = 50$.

7. AB = $\sqrt{8} = 2\sqrt{2}$. BC = $\sqrt{72} = 6\sqrt{2}$. AC = $\sqrt{128} = 8\sqrt{2}$. By definition of "between," B is between A and C, since $2\sqrt{2} + 6\sqrt{2} = 8\sqrt{2}$, and since this assures that the points are collinear.

8. $\overline{AC} \cong \overline{BD}$ since both have lengths $\sqrt{68}$ by the distance formula. $\overline{AC} \perp \overline{BD}$ since the slope of $\overline{AC}$ is 4 and the slope of $\overline{BD}$ is $-\frac{1}{4}$. These are negative reciprocals.

$\overline{AC}$ and $\overline{BD}$ bisect each other since using the midpoint formula each has the midpoint (3,5).

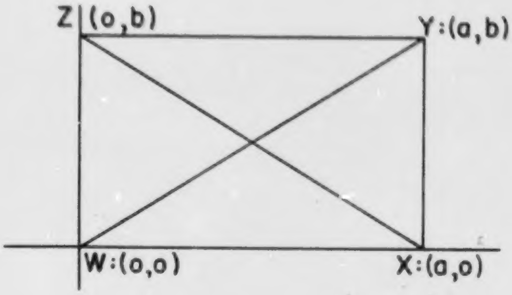
9. a. 7 blocks. b. 5 blocks.

10. a. (a,b).

b. $WY = \sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}.$

$XZ = \sqrt{(0 - a)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}.$

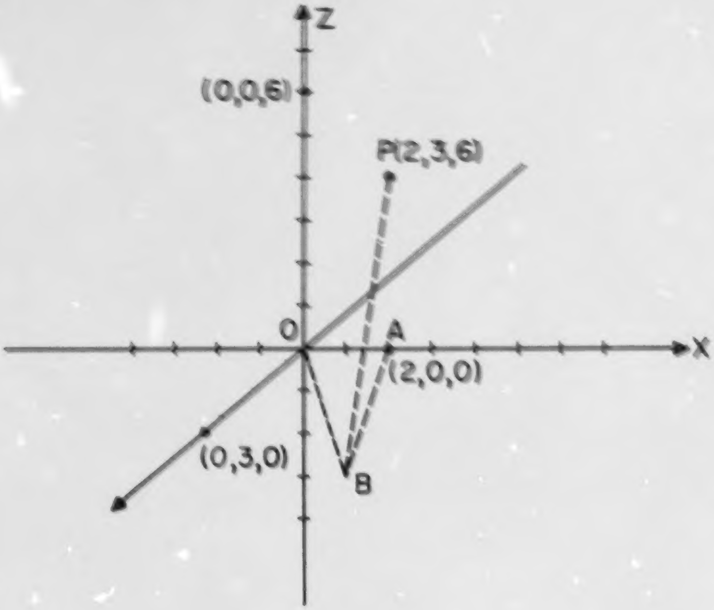
Hence WY = XZ.



11. a. From the meaning of the x, y, and z-coördinates, OA = 2, AB = 3, and BP = 6. By the Pythagorean theorem applied to △ OAB, $OB^2 = 13$, then applied to △ OBP, $OP^2 = 49$ and OP = 7. ($\overline{OP}$ may also be considered a diagonal of a rectangular block.)

b. Generalizing the procedure in part (a), the distance is $\sqrt{x^2 + y_2 + z^2}$.

c. $P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$



Problem Set 16-7.

1. a. (0,6). b. (-2.5,0). c. (2,0). d. (0,0) e. (0,0).

2. a. (8,12).

   b. (-5.5,-1.5).

   c. $\left(\frac{5}{12}, \frac{13}{80}\right)$.

   d. (1.58,1.11).

   e. $\left(\frac{a+b}{2}, \frac{c}{2}\right)$.

   f. $\left(\frac{s}{2}, \frac{r}{2}\right)$.

3. a. (4,2).

   b. $-9=\frac{13+x}{2}$, $30=\frac{19+y}{2}$,

   $x = -31$. $y = 41$.

   The other end point is at (-31,41).

4. The midpoint X of $\overline{AB}$ is (3,2).

   The midpoint Y of $\overline{BC}$ is (-1,3).

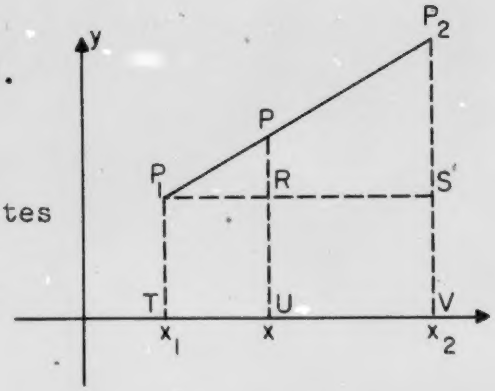
   The midpoint Z of $\overline{CA}$ is (1,0).

By the distance formula $CX = \sqrt{37}$, $AY = \sqrt{52}$ or $2\sqrt{13}$, and $BZ = 5$.

5. By formula, the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ are $W(0,1)$, $X(1,6)$, $Y(4,6)$ and $Z(5,1)$ respectively. $\overline{WX}$ has length $\sqrt{26}$ and slope -5. YZ also has length $\sqrt{26}$ and slope -5. With the same two sides parallel and equal the figure is a parallelogram.

6. By midpoint formulas the other end point of one median is $(\frac{a}{2}, \frac{3a}{2})$, and the other end of the other median is $(\frac{-a}{2}, \frac{3a}{2})$. By the slope formula, the slopes of these medians are 1 and -1. Since 1 is the negative reciprocal of -1, the medians are perpendicular.

7. From the similarity between ΔP_1PR and ΔP_1P_2S, $P_1R = \frac{1}{3} P_1S$. Since $TU = P_1R$ and $TV = P_1S$, $TU = \frac{1}{3} TV$. In terms of coordinates



$$x-x_1 = \frac{1}{3}(x_2 - x_1), \text{ or}$$

$$x = \frac{1}{3}(x_2 - x_1) + x_1 .$$

This can also be written

$$x = \frac{x_2 + 2x_1}{3} .$$

By a similar argument with $\overline{P_1P_2}$ projected on the y-axis,

$$y = \frac{y_2 + 2y_1}{3} .$$

Therefore the coordinates of P are $\left(\frac{x_2 + 2x_1}{3}, \frac{y_2 + 2y_1}{3}\right)$.

8. a. Replacing $\frac{1}{3}$ by $\frac{r}{r+s}$ in the solution of the previous problem we reach

$$x = \frac{r}{r+s}(x_2-x_1) + x_1 ,$$

$$\text{from which,}\quad x = \frac{r(x_2-x_1) + x_1(r+s)}{r+s},$$

$$\text{or}\quad x = \frac{rx_2 + sx_1}{r+s}.$$

By a similar argument, with P_1P_2 projected onto the y-axis,

$$y = \frac{ry_2+sy_1}{r+s}.$$

b. $x = \frac{3\cdot 25+5\cdot 5}{3+5} = 12.5; y = \frac{3\cdot 36+5\cdot 11}{8} = 20.375$.

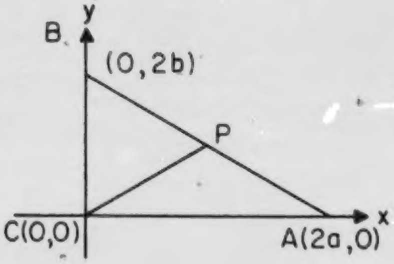
Problem Set 16-8.

1. $DB = \sqrt{(a = 0)^2 + (0 - b)^2} = \sqrt{a^2 + b^2}.$

   $AC = \quad (a - 0)^2 + (b - 0)^2 = \quad a + b^2$.

   Therefore $DB = AC$.

2. Locate the axes along the legs of the triangle as shown.



By definition of midpoint $PA = PB$.

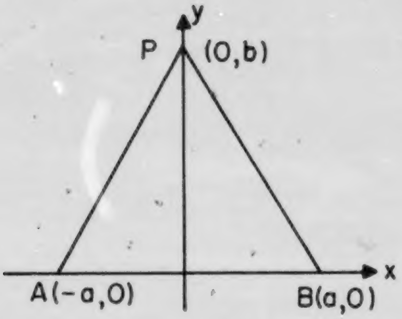
It must be shown that $PA = PC$ (or that $PB = PC$).

$P = (a, b)$. By the distance formula

$$PA = \sqrt{(2a - a)^2 + (0 - b)^2} = \sqrt{a^2 + b^2}$$

$$\text{and } PC = \sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}.$$

3.



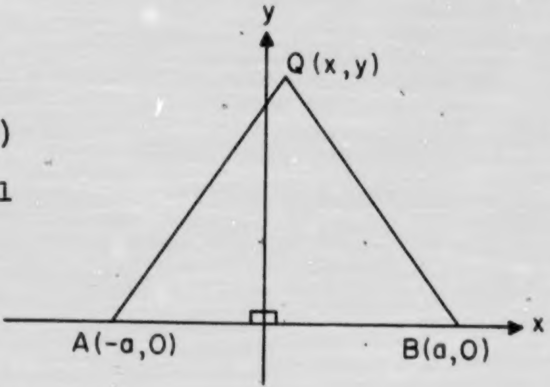
Let the x-axis contain the segment and the y-axis contain its midpoint. Then the y-axis is the perpendicular bisector of the segment. Let P(0,b) be any point of the y-axis, and A(-a,0) and B(a,0) be the endpoints of the segment. Then:

$$PA = \sqrt{(0 + a)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}.$$

$$PB = \sqrt{(a - 0)^2 + (0 - b)^2} = \sqrt{a^2 + b^2}.$$

Hence PA = PB.

4. Place the axes so that the segment will have endpoints A(-a,0) and B(a, 0), and the y-axis will be its perpendicular bisector. Let Q(x, y) be any point equidistant from A and B. From the distance formula



$QA^2 = (x + a)^2 + y^2$ and $QB^2 = (x - a)^2 + y^2$

Since QA = QB, $QA^2 = QB^2$ or

$$(x + a)^2 + y^2 = (x - a)^2 + y^2.$$

Simplifying, $4ax = 0$.

$$x = 0 \quad \text{since } a \neq 0.$$

Hence Q must lie on the y-axis which is the perpendicular bisector of $\overline{AB}$.

5. The midpoint of $\overline{AC} = (\frac{a + b}{2}, \frac{c + 0}{2}) = (\frac{a + b}{2}, \frac{c}{2})$.

The midpoint of $\overline{BD} = (\frac{a + b}{2}, \frac{0 + c}{2}) = \frac{a + b}{2}, \frac{c}{2})$.

Since the diagonals have the same midpoints, they bisect each other.

6.

$R = (\frac{d}{2}, \frac{c}{2})$, $S = (\frac{b + a}{2}, \frac{c}{2})$

Since R and S have the same y-coordinates, $\overleftrightarrow{RS} \parallel \overleftrightarrow{AB}$.

Since $\overline{RS}$ is horizontal,

$$RS = \frac{b + a}{2} - \frac{d}{2} = \frac{b + a - d}{2}$$

$DC = d - b$ and $AB = a$.

Therefore $\frac{1}{2}(AB - DC) = \frac{a - (d - b)}{2} = \frac{b + a - d}{2}$

Hence $RS = \frac{1}{2}(AB - DC)$ which was to have been proved.

7. $R = (2a, 0)$, $S = (2a + 2d, 2e)$

$T = (2b + 2d, 2c + 2e)$, $W = (2b, 2c)$

Midpoint of $WS = (a + d + b, e + c)$.

Midpoint of $TR = (a + b + d, c + e)$.

Therefore $\overline{WS}$ and $\overline{TR}$ bisect each other.

8. Area ΔABC = area (XYBA) + area (YZCB) - area (XZCA).

Area $\Delta ABC = \frac{1}{2}(a+r)(b-a) + \frac{1}{2}(t+s)(c-b) - \frac{1}{2}(r+t)(c-a)$.

Multiplying out and combining terms,

area $\Delta ABC = \frac{1}{2}(rb - sa + sc - tb + ta - rc)$, or

area $\Delta ABC = \frac{a(t-s) + b(r-t) + c(s-r)}{2}$.

9. $ZY^2 = (b-a)^2 + c^2$. $XZ^2 = b^2 + c^2$.

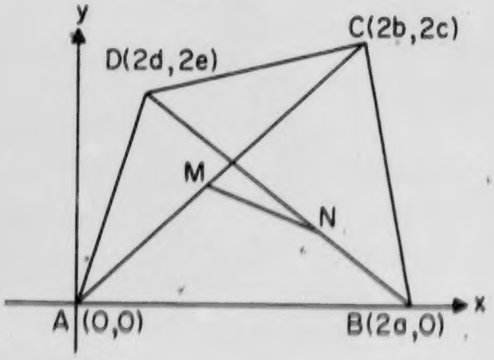
$XY = a$. $XR = b$.

Since $(b-a)^2 + c^2 = (b^2 + c^2) + a^2 - 2ab$,

or $b^2 - 2ab + b^2 + c^2 = b^2 + c^2 + a^2 - 2ab$,

Therefore $ZY^2 = XZ^2 + XY^2 - 2XY \cdot XR$

10. $M = (b,c)$, $N = (A+d,e)$.



$AB^2 = 4a^2$.

$BC = 4(a-b)^2 + 4c^2$.

$CD^2 = 4(b-c)^2 + 4(c-e)^2$.

$DA^2 = 4d^2 + 4e^2$.

$AC^2 = 4b^2 + 4c^2$.

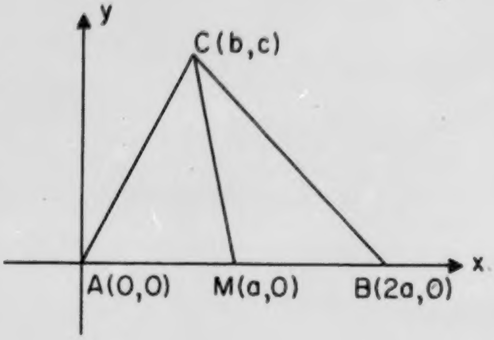
$BD^2 = 4(a-d)^2 + 4e^2$.

$MN^2 = (a+d-b)^2 + (e-c)^2$.

From these expressions the given equation can be verified.

Note that $(a+d-b)^2 = a^2+d^2+b^2+2ad-2ab-2bd$.

11. Place the axes and label the vertices as shown.



$AC^2 = b^2+c^2$.

$BC^2 = (2a-b)^2+c^2$.

$(\frac{1}{2}AB)^2 = a^2$.

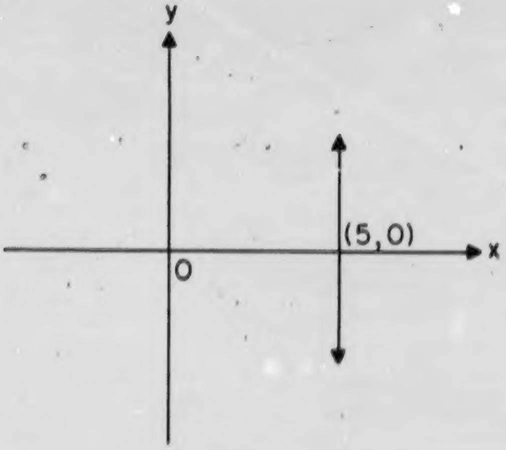
$MC^2 = (a-b)^2 +c^2$.

Since $(b^2+c^2) + (4a^2=4ab+b^2+c^2) = 2a^2+2(a^2-2ab+b^2+c^2)$,

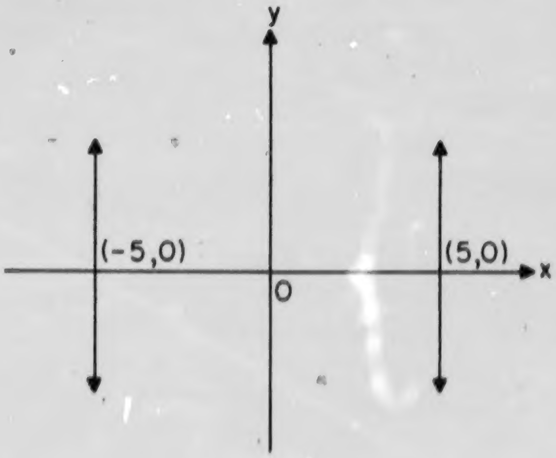
or $4a^2-4ab + 2b^2 + 2c^2 = 4a^2 - 4ab + 2b^2 + 2c^2$

Therefore $AC^2 + BC^2 = (\frac{1}{2}AB)^2 + 2MC^2$.

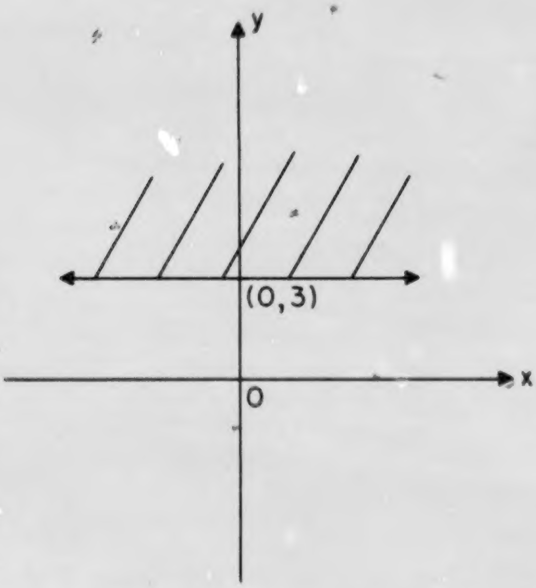
Problem Set 16-9.



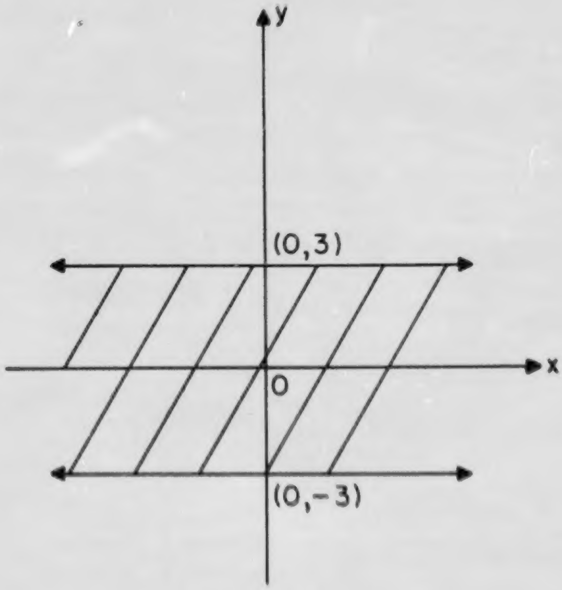
1a. The vertical line through (5, 0).



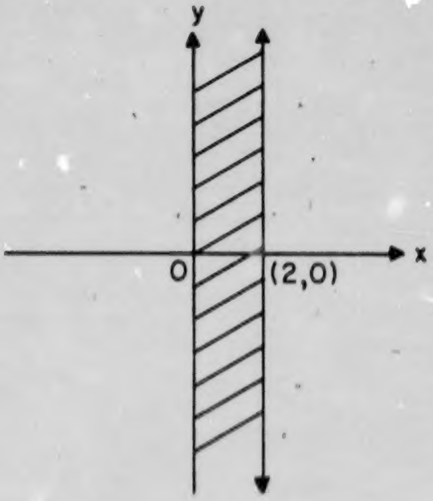
1b. The two vertical lines through (5, 0) and (-5,0).

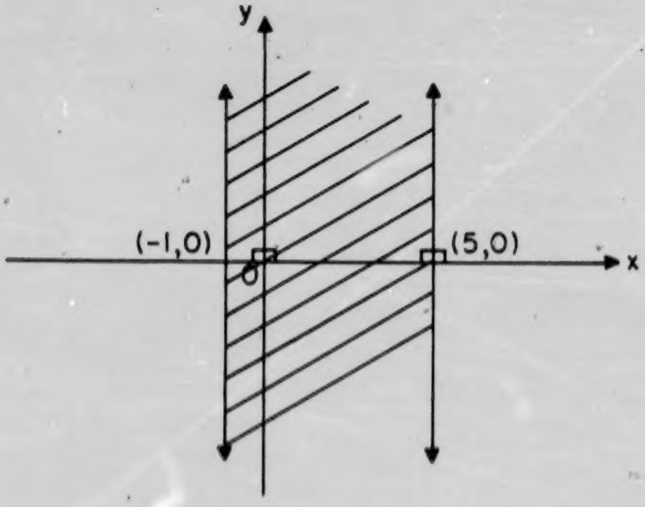


2a. The half-plane above the horizontal line through (0, 3).



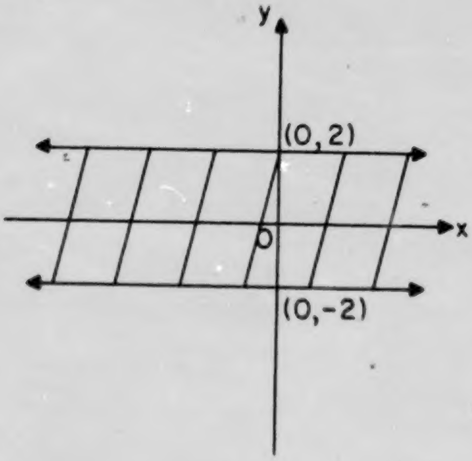
2b. All points between the lines $y = 3$ and $y = -3$.

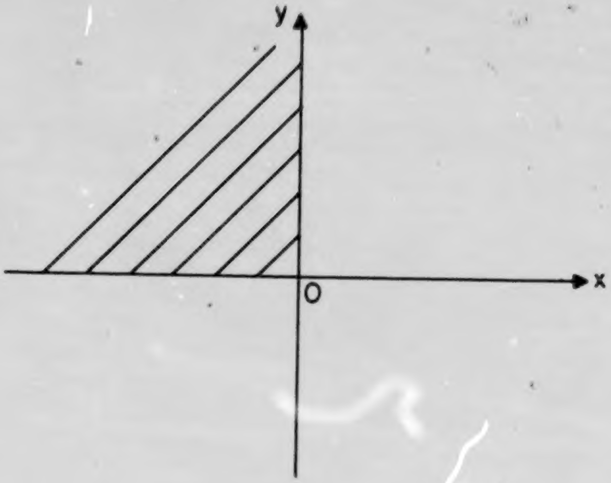




3. All points between the y-axis and the line x = 2.

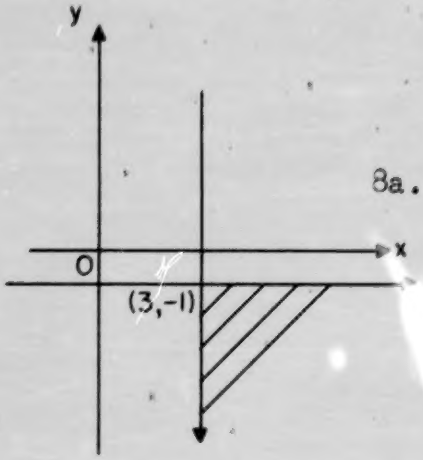
4. All points within or on the boundry of the indicated strip.



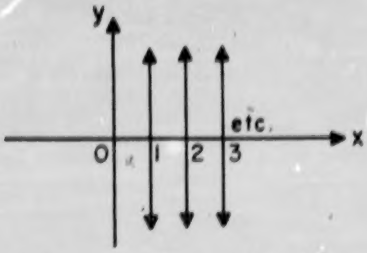


5. All points within, or on the lower boundary of, the indicated strip.

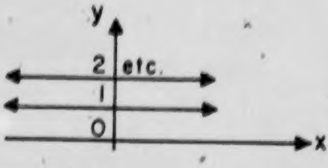
6. All points within the second quadrant.



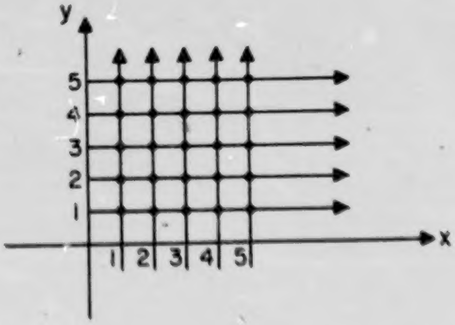
7. All points within indicated angle.



8a. All points on the vertical lines indicated.

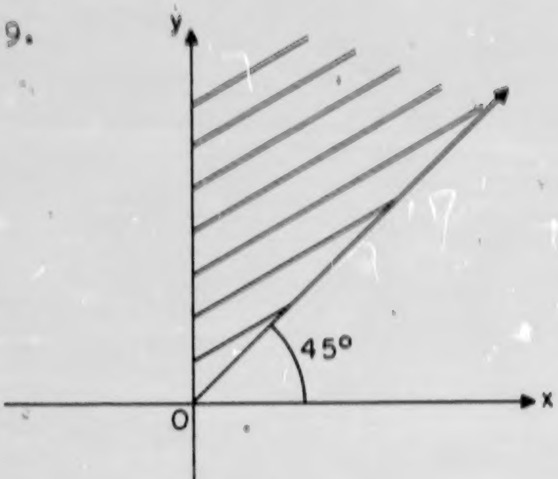


8b. All points on the horizontal lines indicated.



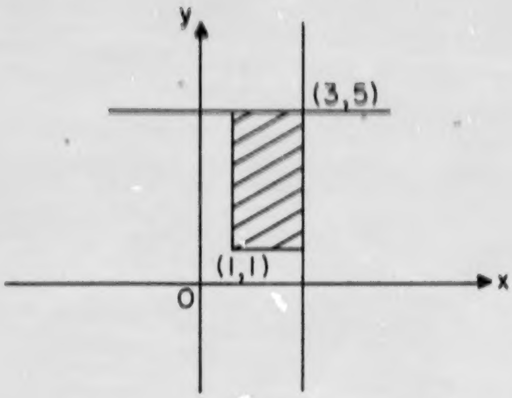
8c. The intersection of the solutions for (8a) and (8b).

9.



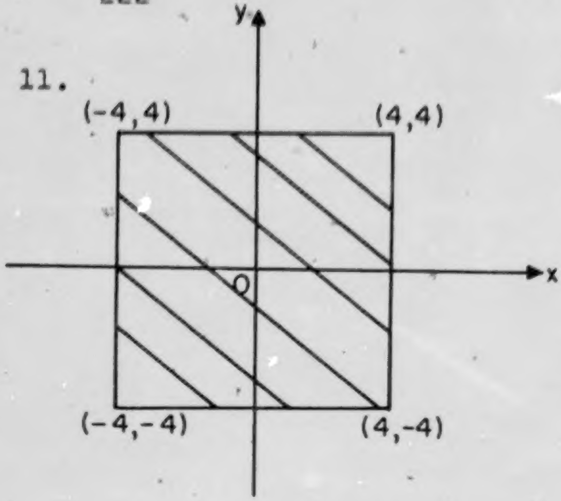
All points within the angle formed by the positive part of the y-axis and the ray from the origin as shown.

10.



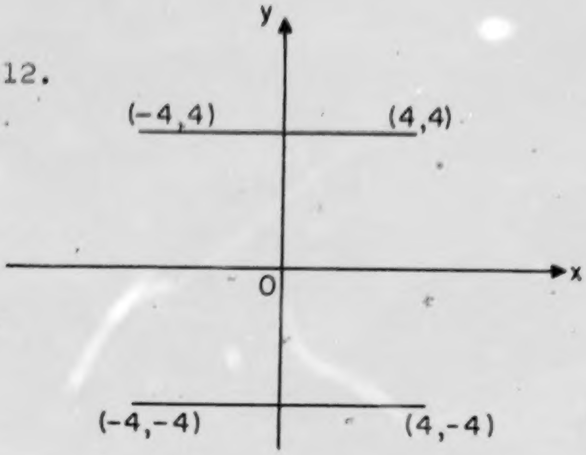
All points within or on the boundary of the indicated rectangle.

11.



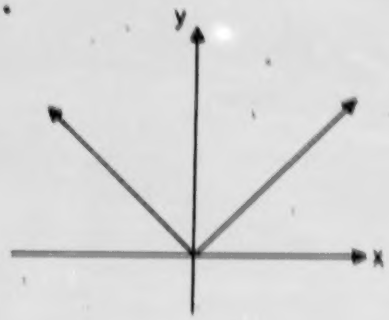
All points in the interior of the square with vertices (4, 4), (-4, 4), (-4, -4), (4, -4).

12.



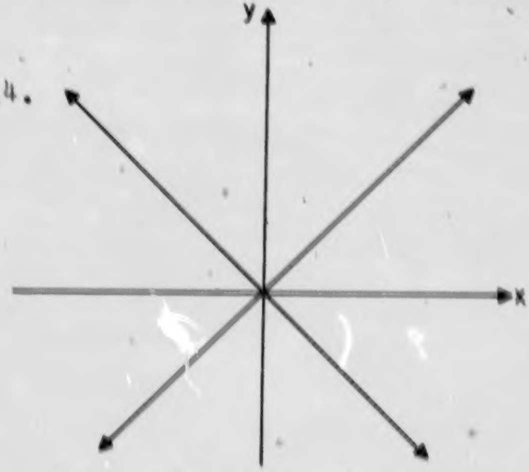
All points on the two segments joining (-4, 4) and (4, 4), and (-4, -4) and (4, -4).

13.

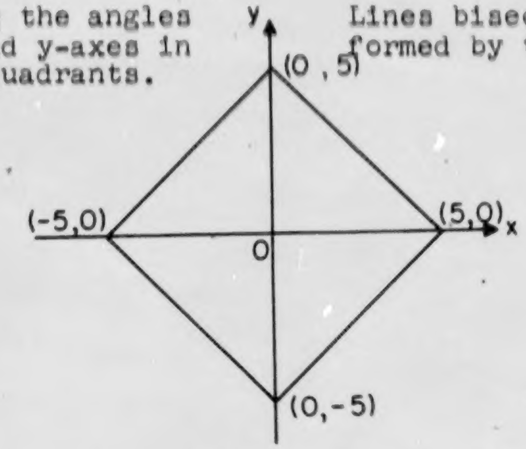


The rays bisecting the angles formed by the x and y-axes in first and second quadrants.

14.



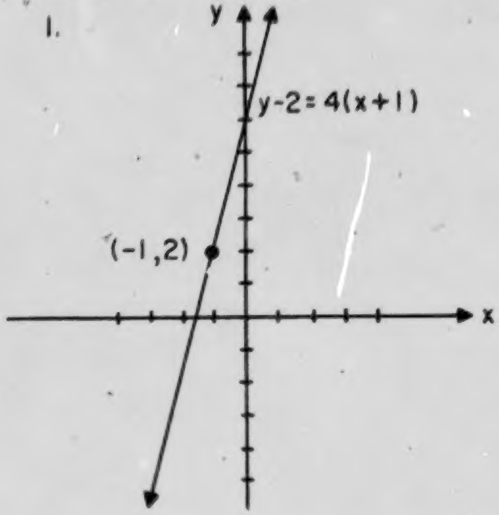
Lines bisecting the angles formed by the x and y-axes.



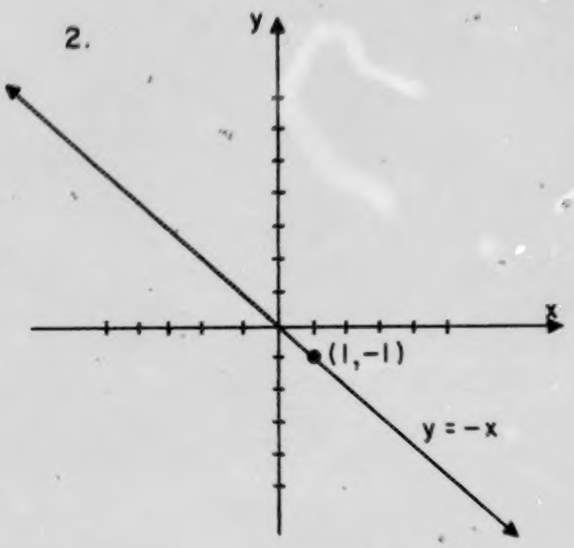
The square with vertices (5, 0), (0, 5), (-5, 0) and (0, -5).

Problem Set 16-10.

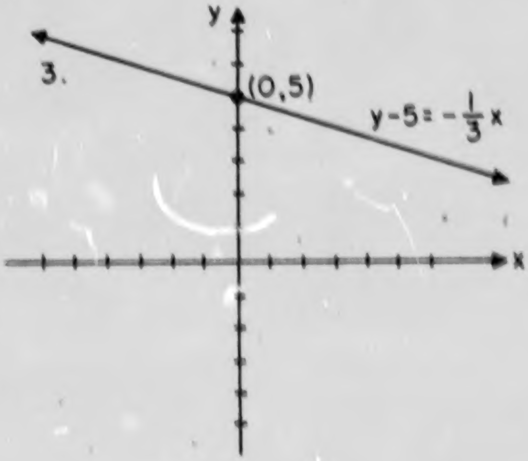
1.



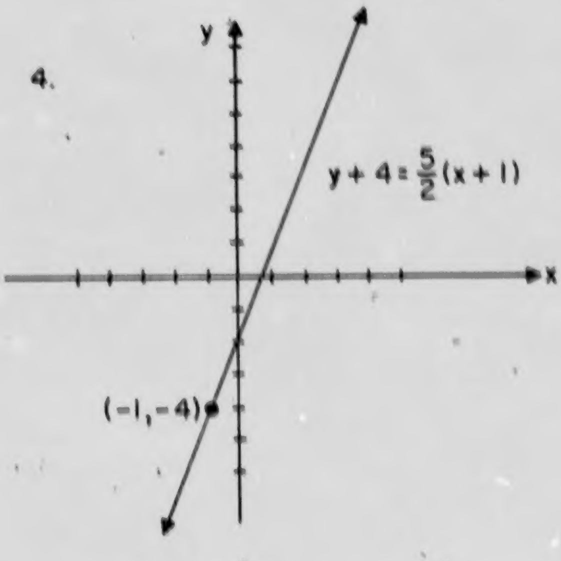
2.



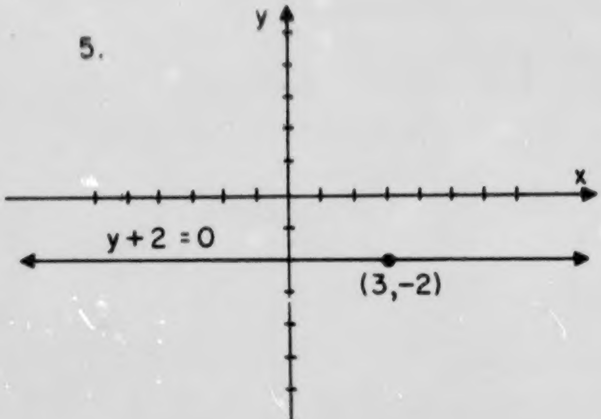
3.



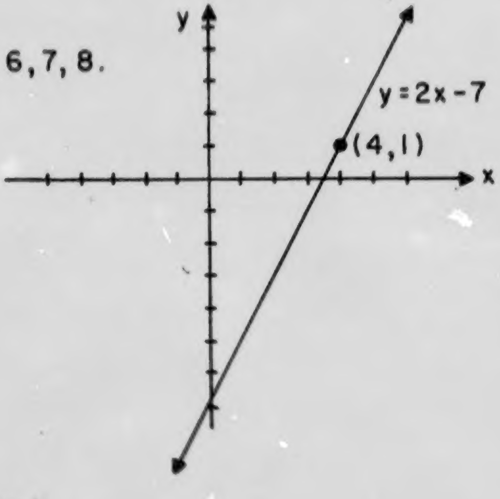
4.



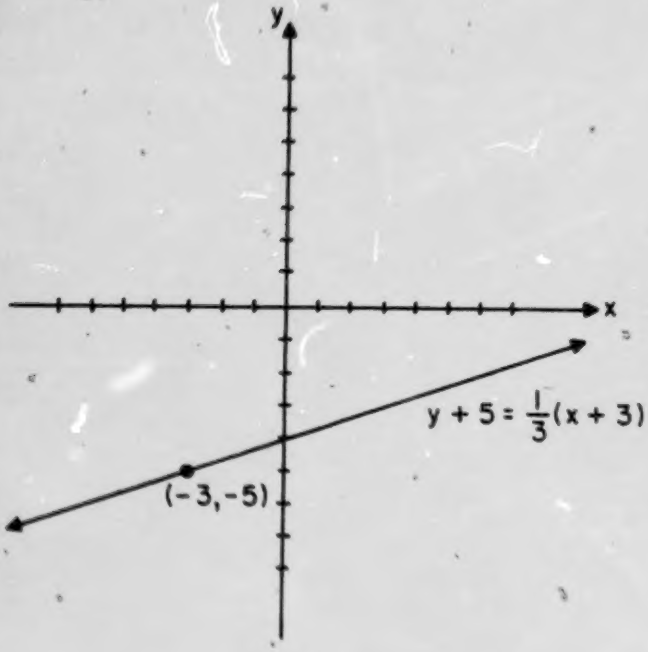
5.



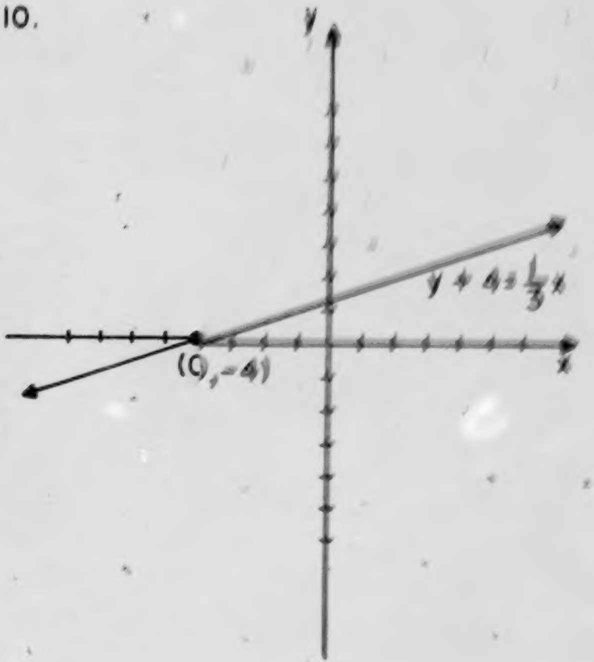
6, 7, 8.



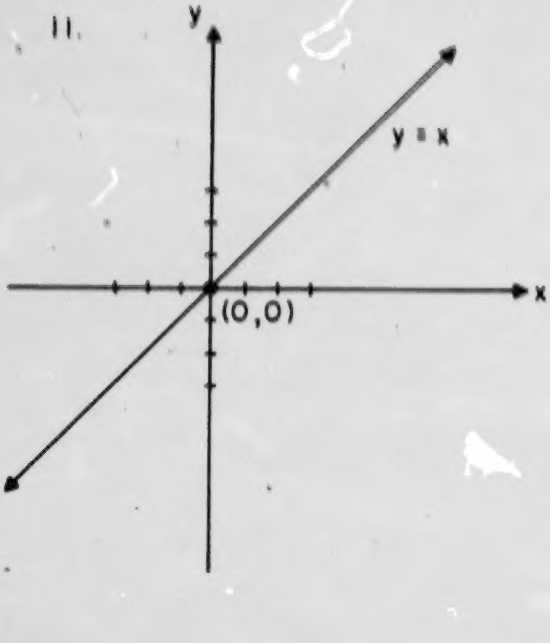
9.



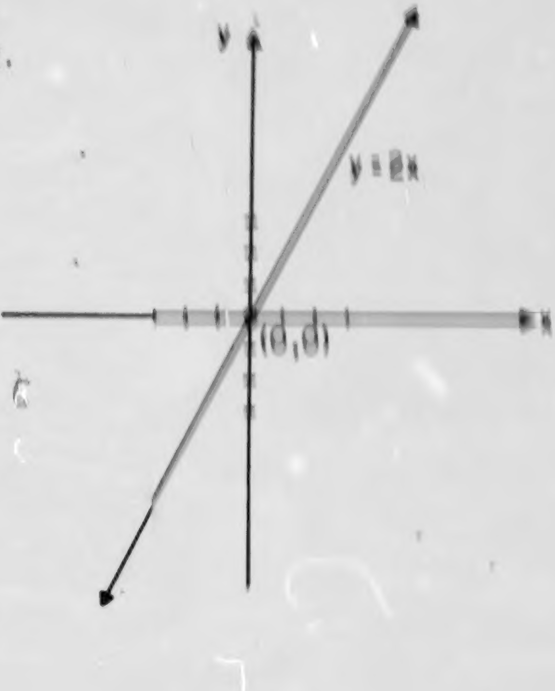
10.



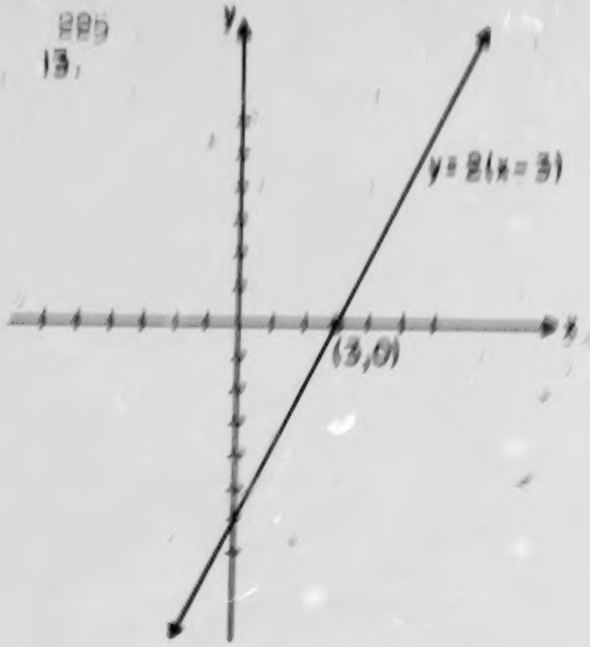
11.



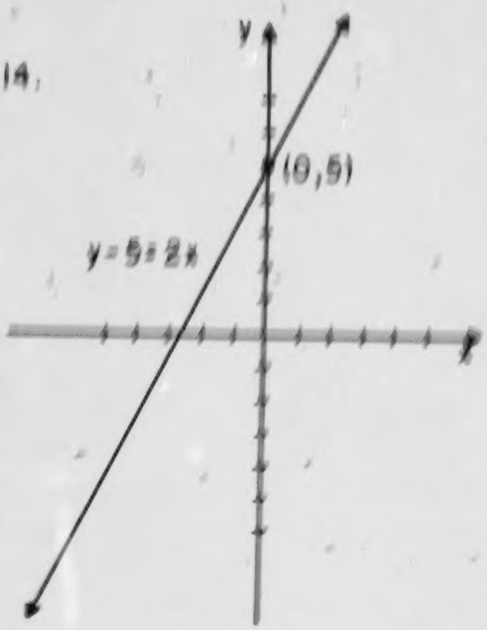
12.



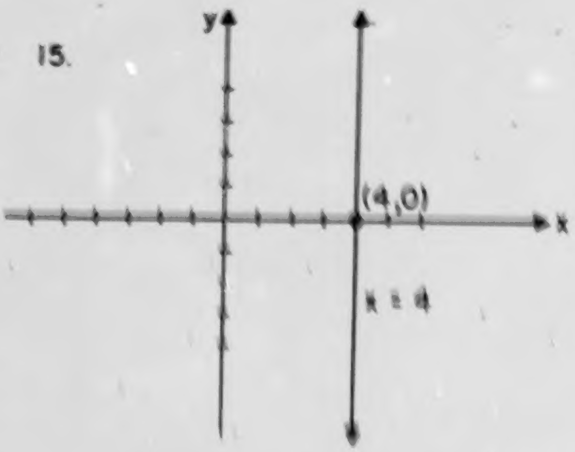
13.


y
x
y = 2(x − 3)
(3,0)

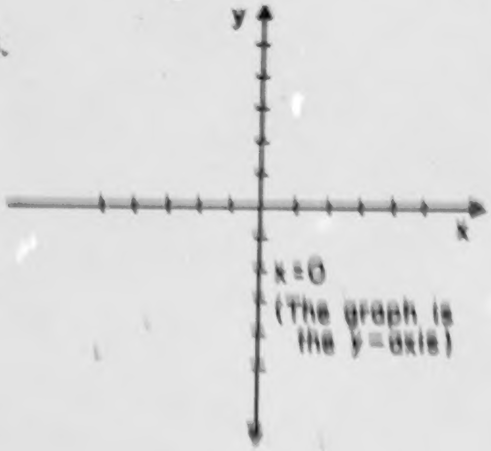
14.


y
(0,5)
y − 5 = 2x
x

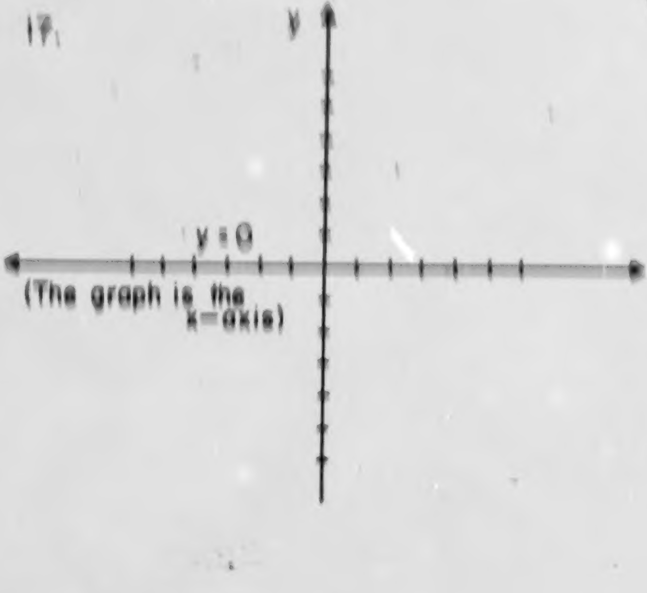
15.


y
(4,0)
x
x = 4

16.


y
x
x = 0
(The graph is the y-axis)

17.


y
y = 0
(The graph is the x-axis)

18. a. The yz-plane.

b. The xy-plane.

c. A plane parallel to the yz-plane and one unit to the right of it.

d. A plane parallel to the xy-plane and two units behind it.

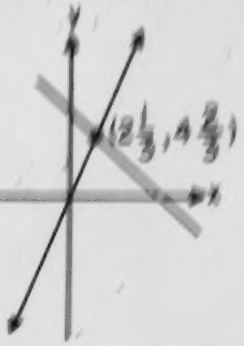
Problem Set 16-12.

1. $y = -\frac{2}{5}x + \frac{7}{5}$. Line with slope $-\frac{2}{5}$, y-intercept $\frac{7}{5}$.
2. $y = 4x - 6$. Line with slope 4, y-intercept -6.
3. $x = -4$. Vertical line through (-4,0).
4. $y = -4$. Horizontal line through (0,-4).
5. $0 = 0$. The whole plane.
6. $0 = 2$. The empty set.
7. The origin, (0,0).
8. The empty set.
9. $y = -\frac{3}{4}x$, $x \leq 0$. Ray from origin, slope $-\frac{3}{4}$, extending into second quadrant.
10. Segment joining (2,5) and (4,10).
11. Line through (0,0) with slope -1.
12. Horizontal line through (0,1).
13. $3x - y - 1 = 0$. $A = 3$, $B = -1$, $C = -1$.
14. $x + y - 1 = 0$. $A = 1$, $B = 1$, $C = -1$.
15. $2x - y - 4 = 0$. $A = 2$, $B = -1$, $C = -4$.
16. $y = 0$. $A = 0$, $B = 1$, $C = 0$.
17. $x = 0$. $A = 1$, $B = 0$, $C = 0$.
18. $y + 3 = 0$. $A = 0$, $B = 1$, $C = 3$.
19. $x + 5 = 0$. $A = 1$, $B = 0$, $C = 5$.

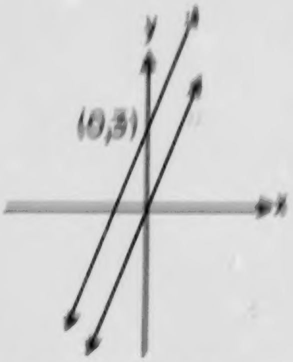
20. $x - 5y = 0$, $A = 1$, $B = -5$, $C = 0$.

Problem Set 16-13.

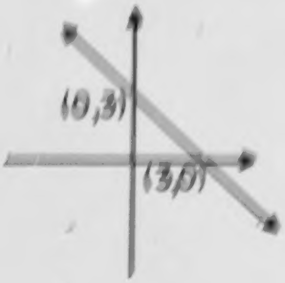
1. a.



b.



c.



One common solution:
$x = 2\frac{1}{3}$, $y = 4\frac{2}{3}$.

No common solution.

The lines are equal. Any two numbers whose sum is 3 is a common solution.

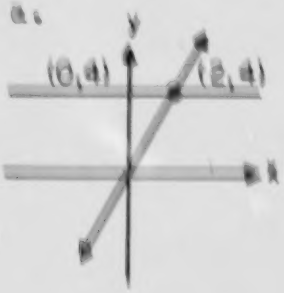
2. a. (1) and (4), (3) and (4).

   b. (1) and (2), (2) and (3), (2) and (4).

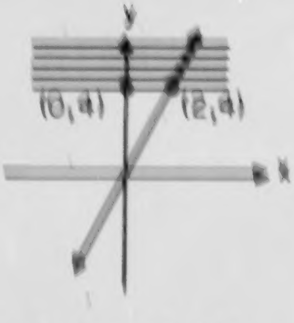
   c. (1) and (3).

3. 4000 miles. ($y = 0$ where the line crosses the x-axis)

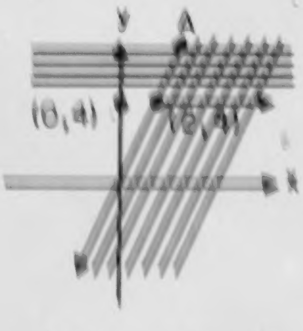
4. a.



b.



c.



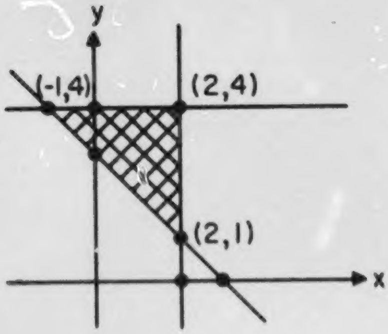
The intersection is point (2,4)

The intersection is the ray shown with end point (2,4).

The intersection is the interior of ∠ABC.

The conditions are $y \angle 2x$ and $y \angle 4$.

5. a.



b. $x + y \quad 3,$

$x \quad 0,$

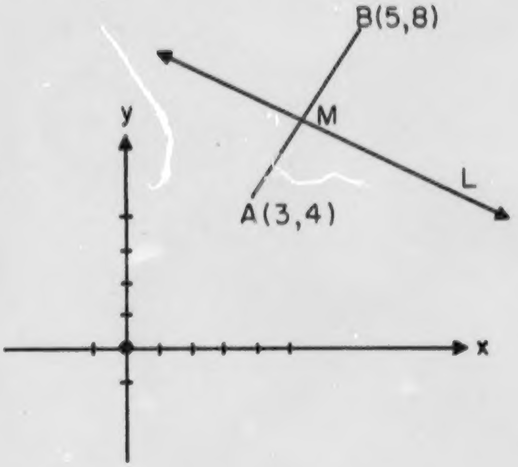
$y \quad 0.$

The intersection is the interior of the triangle with vertices (2,1), (2,4), and (-1,4).

6. The midpoint M has coordinates

$\left(\frac{3 + 5}{2}, \ \frac{4 + 8}{2}\right) = (4,6).$

The slope of AB is $\frac{8 - 4}{5 - 3} = 2$, so the slope L is $-\frac{1}{2}$ and its equation is L:



$y - 6 = -\frac{1}{2}(x - 4),$

$y - 6 = -\frac{1}{2}x + 2,$

$x + 2y = 16.$

Alternate solution L is the locus of points P (x, y) for which PA = PB. This gives $\sqrt{(x - 3)^2 + (y - 4)^2} = \sqrt{(x - 5)^2 + (y - 8)^2}$ which reduces to $x + 2y = 16$.

7. In the preceding problem, we have found the equation

$$L: \ x + 2y = 16$$

Similarly, for M and N we find

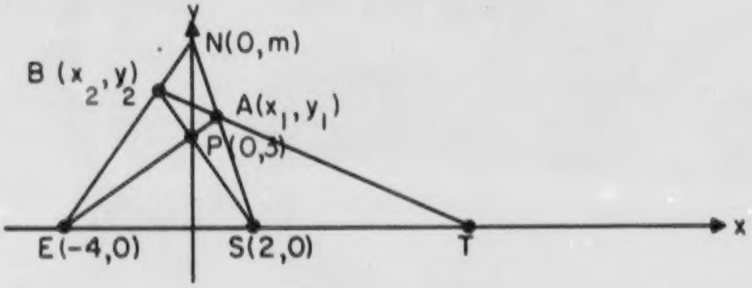
$$M: \ 3x - y = -3,$$

$$N: \ 2x - 3y = -19.$$

The intersection G of L and M is obtained by solving the equations: $G = \left(\frac{10}{7}, \ \frac{51}{7}\right)$

Substituting in the third equation, we find that G lies on N also.

8. Take a coordinate system in which Queen's Road is the x-axis and King's Road is the y-axis.



The coordinates of the elm, spruce, and pine are as indicated. The maple is gone, but its assumed position is labelled (0,m). The slope of $\overleftrightarrow{EP}$ is $\frac{3}{4}$, so its equation (in slope-intercept form) is

$$\overleftrightarrow{EP}: \quad y = \frac{3}{4} x + 3.$$

The slope of $\overleftrightarrow{SM}$ is $-\frac{m}{2}$, so its equation (in point-slope form) is

$$\overleftrightarrow{SM}: \quad y = -\frac{m}{2} (x - 2).$$

Solving these two equations simultaneously, we find the coordinates of A:

$$A: \begin{cases} x_1 = \dfrac{4(m - 3)}{2m + 3}, \\ y_1 = \dfrac{9m}{2m + 3}. \end{cases}$$

Similarly, we get the equations

$$\overleftrightarrow{SP}: \quad y = -\frac{3}{2} x + 3,$$

$$\overleftrightarrow{EM}: \quad y = \frac{m}{4} (x + 4),$$

and the point of intersection is

$$B: \begin{cases} x_2 = -\dfrac{4(m-3)}{m+6}, \\ y_2 = \dfrac{9m}{m+6} \end{cases}$$

The line $\overleftrightarrow{AB}$ has the equation,

$$\overleftrightarrow{AB}: \quad y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1).$$

The intersection T of $\overleftrightarrow{AB}$ and the x-axis is found by putting $y = o$ and solving for x:

$$x = x_1 - y_1\left(\frac{x_2 - x_1}{y_2 - y_1}\right),$$

$$x = \frac{x_1y_2 - x_2y_1}{y_2 - y_1}.$$

Now

$$x_1y_2 - x_2y_1 = \frac{4(m-3)}{2m+3} \cdot \frac{9m}{m+6} + \frac{4(m-3)}{m+6} \cdot \frac{9m}{2m+3}$$

$$= \frac{72m\,(m-3)}{(m+6)\,(2m+3)},$$

$$y_2 - y_1 = \frac{9m}{m+6} \quad \frac{9m}{2m+3},$$

$$= \frac{9m\,(m-3)}{(m+6)(2m+3)}.$$

Dividing, we get $x = 8$. Therefore the treasure was buried 8 miles east of the crossing.

Suppose now that the pine were also missing. Assume coordinates $(0,p)$, for P and carry through the calculation in terms of both m and p. The algebra is a little more complicated, but if it is done correctly both m and p drop out in the final result, which is again $x = 8$.

9. They y-axis is a line through C, perpendicular to the base $\overline{AB}$, i.e., it is the altitude from C. If $\overleftrightarrow{AM}$ is the altitude from A, it has the equation

$$y = m(x + 4),$$

where m is its slope. Since $\overleftrightarrow{AM} \perp \overleftrightarrow{BC}$,

$$m = -\frac{1}{\text{slope } \overleftrightarrow{BC}} .$$

But slope $\overleftrightarrow{BC} = -\frac{8}{7}$, so $m = \frac{7}{8}$, and the equation of $\overleftrightarrow{AM}$ is

$$y = \frac{7}{8}(x + 4).$$

To find the y - intercept, put $x = 0$:

$$y = \frac{7}{8} \cdot 4 = \frac{7}{2} .$$

Now do the same for the altitude $\overleftrightarrow{BN}$.

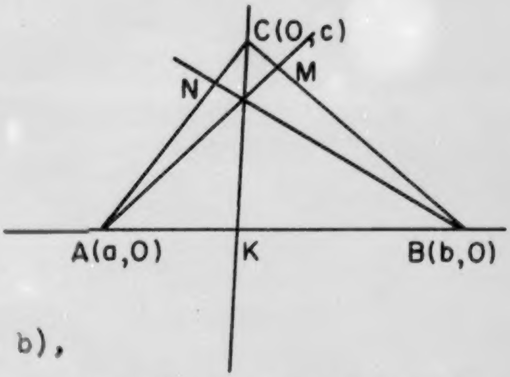
Slope $\overleftrightarrow{AC} = \frac{8}{4} = 2$, so the slope of $\overleftrightarrow{BN}$ is $-\frac{1}{2}$, and its equation is

$$y = -\frac{1}{2}(x - 7) .$$

Putting $x = 0$, we get the Y-intercept

$$y = -\frac{1}{2}(-7) = \frac{7}{2} .$$

Therefore $\overleftrightarrow{AM}$ and $\overleftrightarrow{BN}$ meet at the point $(0, \frac{7}{2})$ on the third altitude $\overleftrightarrow{CK}$.



For the general triangle,

slope $\overleftrightarrow{BC} = -\frac{c}{b}$, slope $\overleftrightarrow{AM} = \frac{b}{c}$,

so $\overleftrightarrow{AM} : y = -(x - a)$,

and the Y-intercept is $-\frac{ba}{c}$.

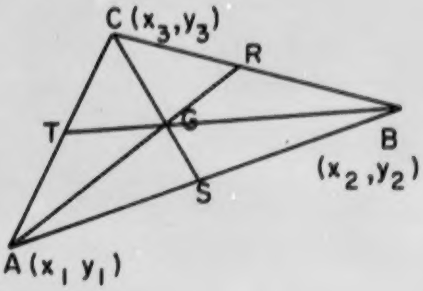
Similarly, slope $\overleftrightarrow{AC} = -\frac{c}{a}$,

slope $\overleftrightarrow{BN} = \frac{a}{c}$, so BN : $y = \frac{a}{c}(x - b)$,

and the Y-intercept is $-\frac{ab}{c}$. Therefore the three altitudes meet at the point $(0, -\frac{ab}{c})$.

10. Let $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C + (x_3, y_3)$.

Then we have



$$R = \left(\frac{x_2 + x_3}{2}\ ,\ \frac{y_2 + y_3}{2}\right)\ ,$$

$$S = \left(\frac{x_1 + x_2}{2}\ ,\ \frac{y_1 + y_2}{2}\right)\ ,$$

$$T = \left(\frac{x_1 + x_3}{2}\ ,\ \frac{y_1 + y_3}{2}\right)\ .$$

The slope of $\overleftrightarrow{AR}$ is

$$m_1 = \frac{\frac{y_2 + y_3}{2} - y_1}{\frac{x_2 + x_3}{2} - x_1} = \frac{y_2 + y_3 - 2y_1}{x_2 + x_3 - 2x_1}$$

If $G = \left(\frac{x_1 + x_2 + x_3}{3}\ ,\ \frac{y_1 + y_2 + y_3}{3}\right)$, then

the slope of $\overleftrightarrow{AG}$ is

$$m_1{}' = \frac{\frac{y_1 + y_2 + y_3}{3} - y_1}{\frac{x_1 + x_2 + x_3}{3} - x_1} = m_1\ ,$$

so G is on the median $\overleftrightarrow{AR}$. Similarly the slope of $\overleftrightarrow{BT}$ is

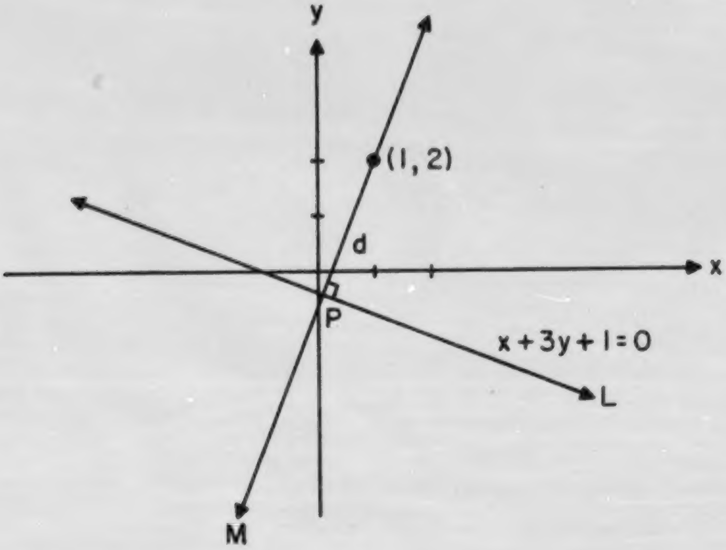
$$m_2 = \frac{\frac{y_1 + y_3}{2} - y_2}{\frac{x_1 + x_3}{2} - x_2} = \frac{y_1 + y_3 - 2y_2}{x_1 + x_3 - 2x_2}\ ,$$

and the slope of $\overleftrightarrow{BG}$ is

$$m_2{}' \quad \frac{\frac{y_1 + y_2 + y_3}{3} - y_2}{\frac{x_1 + x_2 + x_3 - x_2}{3}} = m_2$$

so G is on the median $\overleftrightarrow{BG}$. Similarly we find that G is on the median $\overleftrightarrow{CS}$. Hence the three medians intersect in the point G whose coordinates are the averages of the coordinates of the vertices.

11.



The equation $x + 3y + 1 = 0$ is equivalent to $y = -\frac{1}{3}x - \frac{1}{3}$, which is in slope-intercept form. Therefore the slope is $-\frac{1}{3}$. The line M through (1,2) perpendicular to L has slope 3, so an equation for it is

$$M: \quad y - 2 = 3(x - 1),$$

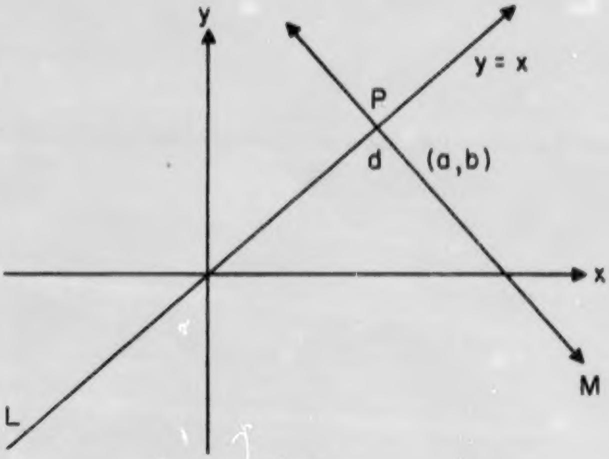
$$y = 3x - 1.$$

Solving the equations for M and L simultaneously to find their intersection P, we get

$$P = \left(\frac{1}{5}, -\frac{2}{5}\right)$$

Computing the distance d from (1,2) to P by the distance formula, we find $d = \frac{4}{5}\sqrt{10}$.

12. The line L with equation $y = x$ has slope 1, so the line M through (a,b) perpendicular to L has slope -1. An equation for M is



$$M: \quad y - b = -(x - a),$$
$$x + y = a + b.$$

Solving for the point of intersection P, we get
$$P = \left(\frac{a+b}{2}, \frac{a+b}{2}\right).$$

The distance is obtained from $d^2 = \left(\frac{a+b}{2} - a\right)^2 + \left(\frac{a+b}{2}\right)^2 = \left(\frac{a-b}{2}\right)^2$,
$$d = \frac{|a-b|}{\sqrt{2}}.$$

From Problem 9 we have $H = \left(0, -\frac{ab}{c}\right)$.

From Problem 10 we have $M = \left(\frac{a+b}{3}, \frac{c}{3}\right)$.

To find D we get the perpendicular bisectors u, v of $\overline{AB}$ and $\overline{BC}$:
$$u: \quad x = \frac{a+b}{2},$$
$$v: \quad y - \frac{c}{2} = \frac{b}{c}\left(x = \frac{b}{2}\right).$$

Therefore
$$D = \left(\frac{a+b}{2}, \frac{c^2+ab}{2c}\right).$$

Now
$$HM^2 = \left(\frac{a+b}{3}\right)^2 + \left(\frac{c^2+3ab}{3c}\right)^2 = \frac{c^2(a+b)^2 + (c^2+3ab)^2}{(3c)^2},$$
$$HD^2 = \left(\frac{a+b}{2}\right)^2 + \left(\frac{c^2+3ab}{2c}\right)^2 = \frac{c^2(a+b)^2 + (c^2+3ab)^2}{(2c)^2},$$
$$MD^2 = \left(\frac{a+b}{6}\right)^2 + \left(\frac{c^2+3ab}{6c}\right)^2 = \frac{c^2(a+b)^2 + (c^2+3ab)^2}{(6c)^2}.$$

From these equations we get

$$HM = 2MD, \quad HD = 3MD,$$
$$HM + MD = HD.$$

This shows that H,M, and D are collinear, that M is between H and D, and that M trisects $\overline{HD}$: $MD = \frac{1}{3} HD$.

Problem Set 16-14.

1. In each case the result is 25. This becomes obvious if radii are drawn to the points on the circle.

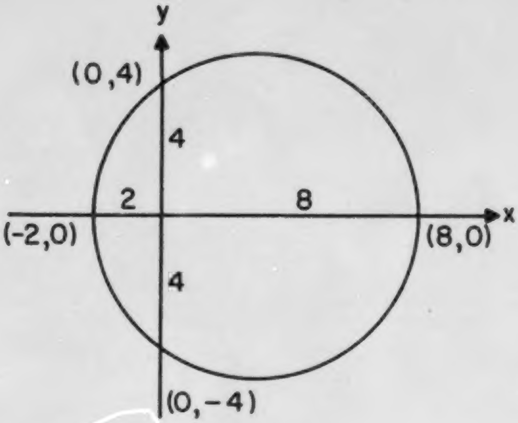
2. a. (1), (3), (4), (6).
   b. (3), (4).
   c. (1).

3. a. Center $(0,0)$; $r = 3$.
   b. $(0,0)$; $r = 10$.
   c. $(1,0)$; $r = 4$.
   d. $(0,0)$; $r = \sqrt{7}$.
   e. $(0,0)$; $r = 2$.
   f. $(4,3)$; $r = 6$.
   g. $(-1,-5)$; $r = 7$.
   h. $(1,0)$; $r = 5$.
   i. $(1,0)$; $r = 5$.
   j. $(-3,2)$; $r = 5$.

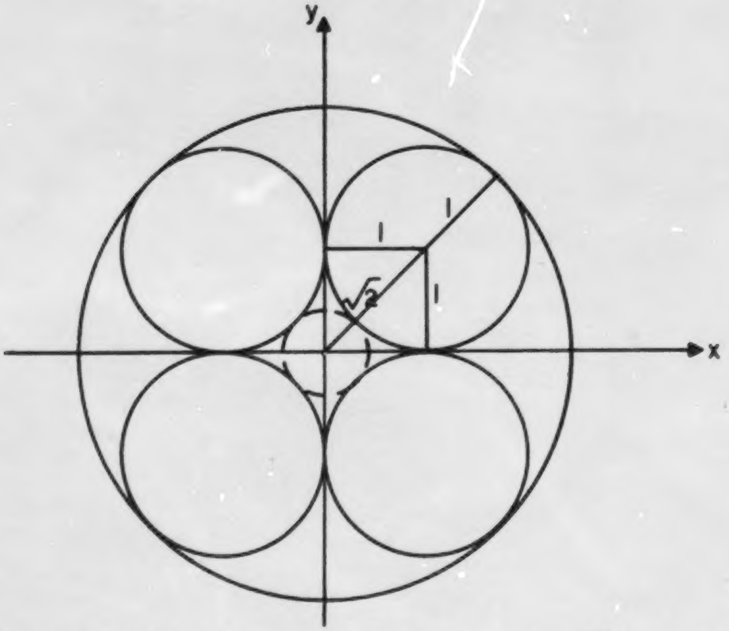
4. a. Replacing x and y in the equation by the given coordinates satisfies the equation.
   b. $x^2 - 10x + y^2 = 0$,
   $(x^2 - 10x + 25) + y^2 = 25$,
   $(x - 5)^2 + 7y - 0)^2 = 5^2$.
   The center of the circle is $(5,0)$; the radius is 5.
   c. The ends of the diameter along the x-axis are $(0,0)$ and $(10,0)$. The slope of the segment joining $(0,0)$ and $(1,3)$ is 3. The slope of the segment joining $(10,0)$ and $(1,3)$ is $-\frac{1}{3}$. Since 3 and $-\frac{1}{3}$ are negative reciprocals, the lines are perpendicular and a right angle is formed.

5. a. The x-axis intersects the circle where $y = 0$, that is where $(x - 3)^2 = 25$, or at points $(-2,0)$ and $(8,0)$. The y-axis intersects the circle where $x = 0$, that is where $9 + y^2 = 25$, or at points $(0,4)$ and $(0,-4)$.

b. $2 \cdot 8 = 4 \cdot 4 = 16$.



6.

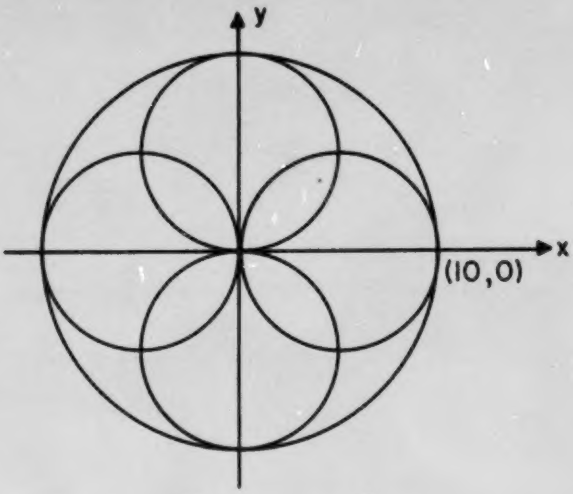


The radius of the larger circle is $1 + \sqrt{2}$. So the equation is

$$x^2 + y^2 = (1 + \sqrt{2})^2$$

There would be another tangent circle of radius $\sqrt{2} - 1$ and the same center.

7.



The including circle is $x^2 + y^2 = 100$.

8. a. $y = m(x + 7)$.

b. $x^2 + \{m(x + 7)\}^2 = 16$,

$$(1 + m^2)x^2 + 14m^2x + (49m^2 - 16) = 0,$$

$$x = \frac{-14m^2 + \sqrt{(14m^2)^2 - 4(1 + m^2)(49m^2 - 16)}}{2(1 + m^2)}$$

$$= \frac{-14m^2 + \sqrt{4(16 - 33m^2)}}{2(1 + m^2)}$$

$$= \frac{-7m^2 + \sqrt{16 - 33m^2}}{1 + m^2},$$

$$y = m(x + 7) = \left(\frac{-7m^2 + \sqrt{16 - 33m^2} + 7 + 7m^2}{1 + m^2} \right) m$$

$$= \frac{m(7 + \sqrt{16 - 33m^2})}{1 + m^2} .$$

If $16 - 33m^2 > 0$, there are two points of intersection:

$$P_1 = \frac{-7m^2 + \sqrt{16 - 33m^2}}{1 + m^2}, \quad \frac{m(7 + \sqrt{16 - 33m^2})}{1 + m^2}$$

$$\frac{-7m^2 - \sqrt{16 - 33m^2}}{1 + m^2}, \quad \frac{m(7 - \sqrt{16 - 33m^2})}{1 + m^2} .$$

If $16-33m^2 = 0$, there is one point of intersection:

$$P = \left(\frac{-7m^2}{1 + m^2} \;,\; \frac{7m}{1 + m^2}\right) .$$

If $16-33m^2 < 0$, there is no point of intersection.

c. From part b., $m^2 = \frac{16}{33}$, $m = \pm \frac{4}{\sqrt{33}}$.

This means that the two lines

$$y = \frac{4}{\sqrt{33}} \quad (x + 7),$$

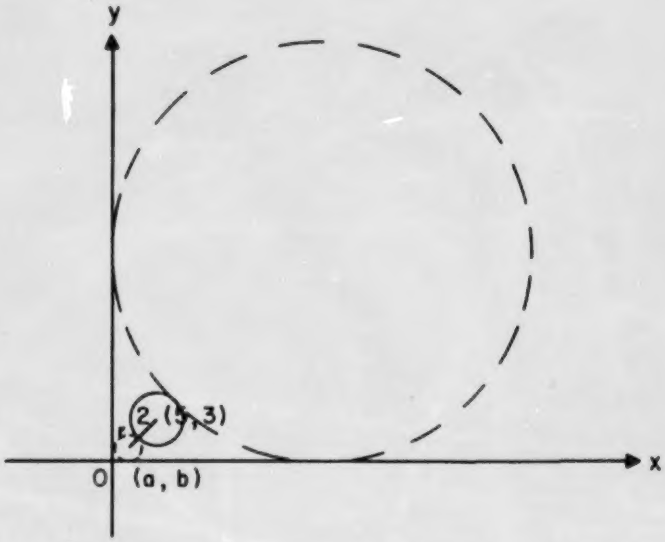
$$y = -\frac{4}{\sqrt{33}} \quad (x + 7)$$

are tangent to the circle.

9. Put the given equation in standard form

$$(x - 5)^2 + (y - 3)^2 = 2^2 .$$

The given circle has center (5,3), radius 2.



Let the required circle have center (a,b), radius r. Then $b = a = r$ and the distance from (a,b) to (5,3) is $a + 2$. That is,

$$a + 2 = \sqrt{(5 - a)^2 + (3 - a)^2}\ ,$$

$$a^2 + 4a + 4 = 2a^2 - 16a + 34,$$

$$a^2 - 20a + 30 = 0,$$

$$a = \frac{20 \pm \sqrt{400 - 120}}{2} \qquad \frac{20 \pm \sqrt{200}}{2}$$

$$a = 10 \pm \sqrt{70}.$$

Thus there are two solutions:

$$(x - a)^2 + (y - a)^2 = a^2\ ,$$

$$x^2 + y^2 - 2ax - 2ay + a^2 = 0,$$

where $a = 10 + \sqrt{70}$ or $a = 10 - \sqrt{70}$.

Review Chaps. 13-16.

1. 1.
2. 1.
3. 0.
4. 1.
5. 0.
6. 0.
7. 0.
8. 1.
9. 1.
10. 0.
11. 0.
12. 0.
13. 0.
14. 1.
15. 0.
16. 1.
17. 1.
18. 0.
19. 1.
20. 1.
21. 0.
22. 0.
23. 1.
24. 1.
25. 0.
26. 1.
27. 0.
28. 1.
29. 1.
30. 0.
31. 0.
32. 1.
33. 1.
34. 1.
35. 1.
36. 1.
37. 0.
38. 0.
39. 1.
40. 0.
41. 1.
42. 0.
43. 0.
44. 0.
45. 1.
46. 0.
47. 1.
48. 0.
49. 1.
50. 0.

GEOMETRY

Teacher's Manual

Part III

Talks to Teachers

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Facts and Theories.

Science today is playing an increasingly important part in the life of the individual. No one can claim to be truly educated unless he has a reasonable understanding of the facts and methods of science. This does not mean that we must all become nuclear physicists, nor that we must spend all our time reading books and attending lectures on the latest collection of particles discovered by the physicists. But it does impose on us the obligation to learn enough of the facts of modern science to provide a foundation for understanding. It does imply an intelligent selection of material to be learned.

We, as educators, are especially obligated to make such a selection for our students. They come to us with a miscellaneous hodgepodge of disjointed facts and pseudo-facts, gleaned from newspapers, magazines, books, and other sources. We must help them -- with our own limited information -- to straighten out their ideas, to build a reasonable conceptual structure upon which they can hang new facts, to distinguish between that which is significant and that which is not, and, perhaps most important of all, to understand how new knowledge is acquired. If pursued to the extreme, this last goal would lead us to the far reaches of epistomology and scientific method, which have been the subjects of many weighty tomes written by scholars over many lifetimes, and about which the last word has certainly not been uttered. But to dismiss this topic entirely as being too subtle for the immature minds of our students is to deny them the opportunity of becoming a little more mature in our classrooms.

What should be the aims of the mathematics teacher, in the light of what we have just said?

Certainly we should help the student to become acquainted with the facts of mathematics by working with them. We agree that our subject is an essential tool in science and in daily life, and that the student should acquire a working facility in it. Therefore we teach him arithmetic, elementary algebra, intuitive geometry in the lower grades, advanced algebra, synthetic and analytic geometry, possibly calculus and other topics in the higher grades.

It would be difficult, however, to defend the teaching of all these subjects on the grounds of utility alone. No one pretends, for example, that it is of practical importance that the bisector of an angle of a triangle divides the opposite side in the way that it does. We proceed, then, to the second aim, of developing in the student an appreciation of clear, logical reasoning as exemplified in mathematics, and an ability to transfer this type of reasoning to other situations. We have been moderately, though not eminently, successful in this respect in the past. Whether our present efforts will tend to further this objective remains to be seen. We certainly hope so.

A third aim, which has been receiving more attention of late, is to develop in the student an understanding of the structure of mathematical systems. We are beginning to speak of closure, commutativity, distributivity and so on in dealing with number systems, and -- still too timidly, perhaps -- of the axiomatic nature of geometry.

This third aim is closely related to the broader one mentioned earlier, of helping the student to understand how new knowledge is acquired, how man learns about the physical world, how he constructs, develops and tests theories about the physical, biological, social, and economic aspects of life around him. Let us address ourselves briefly to these questions.

Whether we recognize it or not, theory plays an indispensable role in our study of any field whatsoever. The acts of naming, classifying, and generalizing are conceptual in nature. Even emotional reactions to stimuli depend on a structuring of experience. The real world -- whatever that may mean -- reaches us only through our senses, and we can study it and think about it only by constructing a conceptual world to correspond to it. In setting up a particular discipline, it is not necessary, however, to refer back always to the primary data supplied by our senses. The raw material for a theory at one stage may be the conceptual world of a previous stage. For example, the classical geometry of various surfaces in three dimensions may be taken as the jumping-off place for a study of abstract metric spaces, and we would then abstract from this classical geometry, testing our new theory against it. In every case, then, we operate simultaneously in two different "planes". One is the primary, intuitive plane, containing the raw data from which our theory will be abstracted. This, following Bridgman, we call the "P-plane". The second is the conceptual plane, the "C-plane". Initially, the C-plane is empty, waiting to be filled with the concepts and relations which we construct.

We have complete freedom with respect to the concepts and relations which we choose to insert in the C-plane, so long as we do not assert any connection between it and the P-plane. Naturally, we hope eventually to set up a correspondence between the two planes, and this hope guides our constructions and our choice of language. Logically, there is no necessity to make the language in the C-plane correspond to that of the P-plane, and in order to avoid confusion it might be better to use different terms entirely. For example, the "points", "lines", and "planes" of axiomatic geometry (the C-plane) might be replaced by other terms which have not been preempted in physical geometry (the P-plane). But once the formal distinction between the two planes and their languages has been established and understood, there is a psychological advantage to be gained from the use of the same terms, for the proposed correspondence is then transparently indicated. Thus, we know that the geometrical "point" is meant to correspond to the physical point, the geometrical "line" to the physical line, and so on. We can intuit, conjecture, and then perhaps prove theorems in the C-plane by peeking over into the P-plane at the corresponding "facts", arrived at by experiment there. For example, the concurrence of the medians of a triangle could be guessed from drawing a number of physical triangles and their medians on a piece of paper. This type of experience is extremely valuable and constitutes an important psychological adjunct to mathematical discovery. It must be pointed out carefully, though, that formal proof in the C-plane is necessary. Furthermore, the logical conclusion to be drawn from this combined guessing and proving process is not that we have made the geometrical theorem more certain by experimental verification. The truth of the theorem has been established (in the C-plane) with complete certainty by logical deduction from the axioms. Rather, our feeling of satisfaction on seeing that the theorem works out on paper should stem from the confirmation of the correspondence between the two planes. What we do tend to establish by such empirical tests is the adequacy of our postulate system to bring about a close correspondence.

Consider, for example, what our situation would be if we had in our system all of the postulates of Euclidean geometry except for the parallel postulate. Suppose, then, that we measured the angles of many triangles and found, within the limits of experimental error, that the sum of the measures of the angles was 180. Then, passing to the C-plane, we attempted to prove the corresponding result as a theorem, and of course failed. The correct conclusion to draw would be that (a) we were not clever enough to find a proof, or (b) that our axiom system was not adequate for the purpose. Historically, it was the belief that (a) was the only possibility, together with an imperfect understanding of axiomatics, that delayed the development of non-Euclidean geometry. Eventually, of course, this very problem led to our present deeper understanding of the connection between fact and theory.

What are the considerations that govern our choice of undefined elements and relations and unproved propositions (axioms, postulates)? Certainly we want our system to be consistent: a proposition and its contradiction should not both be provable in the system. If we regard our axioms as inputs and our theorems as outputs, then economy and fruitfulness are desirable as increasing output per unit input. Of course, this analogy is not to be taken too seriously, but it indicates why we should not postulate everything. Unfortunately, some geometry texts nowadays go to the extreme of setting down fifty or more postulates. There is nothing logically wrong with this, but it militates against economy, elegance, intuitiveness, simplicity, and ease of verification in a particular interpretation -- properties that are certainly desirable.

One property that we have not mentioned is that of being categorical. This means that every two concrete interpretations (models) of the system will be essentially the same: it is possible to set up a one-to-one correspondence between the elements and relations of the two interpretations, so that they may be regarded as identical except for the names assigned to the elements and relations. The two models are then said to be isomorphic. If we start with a particular P-plane and wish to describe it completely by means of an axiom system, without permitting any non-isomorphic models, then we try to make our system categorical. This is the case with Euclidean geometry or the real number system.

Sometimes we reap an unexpected harvest from the construction of a categorical system. We may find two apparently different interpretations, and can then conclude that they are essentially identical because the system is categorical. Any theorem which holds in one model is then sure to hold in the other. An example is the pair of models M_1, consisting of the real numbers under addition, and M_2, consisting of the positive real numbers under multiplication. The one-to-one correspondence $M_1 \longleftrightarrow M_2$ is established by the exponential function (from M_1 to M_2) and the logarithm (from M_2 to M_1). Another example is the pair of physical processes, diffusion of a gas and heat-flow, both being governed by the same differential equation. Still another example is the isomorphism of Euclidean plane geometry with the collection of all real-number pairs. This isomorphism allows us to solve geometrical problems by means of algebra, and vice versa.

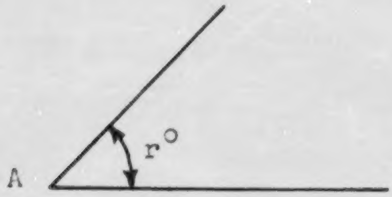
At other times, we find it more profitable to make our system non-categorical. This is true when we have several P-planes which bear some resemblance to each other. If we can construct a suitable C-plane so that each of the P-planes is an interpretation of it, then anything we prove in the C-plane will hold in all of its non-isomorphic models. This happens, for example, in the case of group theory. It also happens when we state a few, but not all of the axioms of Euclidean geometry. In this case our theorems, being provable, say, without the parallel postulate, must hold also for all geometries satisfying the stated axioms. There is no reason to hide this desirable state of affairs from our students, for fear of violating their intuitions about space. Rather, we should regard such occasions as valuable opportunities for teaching an important lesson.

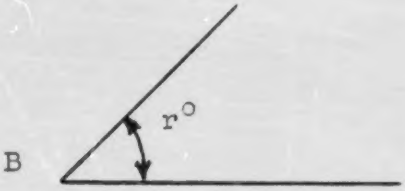
Our discussion here has been far from exhaustive. We hope that it has served the purpose of pointing to a desirable and sometimes neglected goal in education, and that it has indicated how we, as teachers of mathematics, can approach this goal.

8.

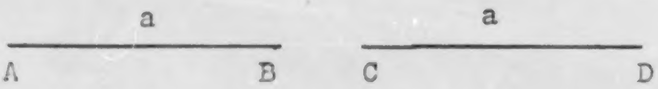
Equality, Congruence, and Equivalence.

1. Angles and segments. In describing the relation of "equality" between angles and segments, this book departs from common usage. Before explaining why this has been done, let us first note quickly how the new usage compares with the old. Suppose we have two angles with the same degree measure r, like this





and two segments of the same length, like this:



In these two instances, the facts are plain. They would be reported in the following ways, in the old and new terminologies.

In Words		In Symbols	
Old	New	Old	New
The angles are equal.	The angles are congruent.	$\angle A = \angle B$	$\angle A \cong \angle B$ (or $m\angle A = m\angle B$)
The segments are equal.	The segments are congruent.	$\overline{AB} = \overline{CD}$	$\overline{AB} \cong \overline{CD}$ (or $AB = CD$).

From the table it is plain that the new usage is not complicated. We have simply substituted one word for another, and one symbol for another. Of course, even simple changes should be made only for good reasons; they go against everybody's habits, and cause much more trouble at first than their simplicitly would suggest. We believe that there are very good reasons for the use that we have made of the word congruence. Following is an explanation of what these reasons are.

2. Various kinds of equality. The word "equals" is commonly used in mathematics in at least this many different senses:

(1) When we write $2 + 4 = 3 + 3$, we mean that the number $2 + 4$ and the number $3 + 3$ are exactly the same number (namely, 6.) Here "equals" means "is the same as".

(2) When we say that two angles are equal, we mean that they have the same measure, or the same shape.

(3) Two circles are equal if they have the same radius.

(4) Two segments are equal if they have the same length.

(5) Two triangles are equal if they have the same area.

(6) Two polyhedrons are equal if they have the same volume.

These uses of "equals" divide sharply into three groups.

(I) The first meaning ("is the same as") stands entirely alone. This is the logical identity. It arises in all branches of mathematics, including geometry.

(II) "Equality" expresses the same basic idea for angles, circles, and segments, in (2), (3), and (4). It means in each case that the first figure can be moved so as to coincide with the second without stretching. (For a fuller explanation, see appendix VI, on Rigid Motion). This idea is geometric, and is one of the most basic ideas in geometry. Applied to triangles, it is always described as congruence and not as equality.

(III) "Equality" to mean equal areas or equal volumes, as in (5) and (6), implies that two things are equal if they contain the same amount of "stuff".

These are the three main ideas involved. We notice that the words and the ideas overlap both ways. Not only is the word "equals" used in two widely different senses, but the basic idea involved in (2), (3), and (4) is expressed by two apparently unrelated words.

Obviously students can and do learn to keep track of what is meant, even when the words and the ideas overlap in this way. All of us learned to do this, when we were in the tenth grade. The whole thing becomes easier to learn, however, and easier to keep track of, if the words match up with the ideas in a simpler and more natural way. This can be done as follows:

(I) We can agree to write " = ", and say "equals", only when we mean "is the same as". (This is the standard usage in nearly all of modern mathematics).

(II) We already have a word to express the idea that one triangle can be made to coincide with another; we say that they are congruent. We can use the same word to express the same idea when we are talking about angles, circles or segments.

(III) When we want to convey the idea that two triangles have the same area, we can simply say that they have the same area.

Notice that if we do this we have not introduced any new words into the language of geometry. We are not trying to be technical. All that we are trying to get at is a situation in which the familiar and available words correspond in a natural way to the familiar and basic ideas. The correspondence looks like this:

(I) =, between any two things whatever, means "is the same as".

(II) $\cong$, between any two geometric figures whatever, means that one can be moved so as to coincide with the other.

(III) Equality of area, equality of volume, and so on, are to be described explicitly as such.

All this is straightforward language. We believe that your students will find it easy to learn and easy to use.

3. Equivalence relations. All the uses of "equals", in mathematics or otherwise, involve the notion of two things being alike in some respect. The particular respect to be considered may be made explicit, as in usage (5) above, or it may not, as in "All men are created equal". As mentioned above, mathematicians have pretty generally agreed to use the word to mean "alike in all respects"; that is, identical. Instead of the other usage they speak of an "equivalence relation". A relation between pairs

of objects, from some given set, is called an equivalence relation if it has the following three properties:

(1) It is reflexive. That is, any object of the set is equivalent to itself.

(2). It is symmetric. That is, if A is equivalent to B, then B is equivalent to A.

(3) It is transitive. That is, if A is equivalent to B, and B is equivalent to C, then A is equivalent to C.

In a mathematical development we may use several different kinds of equivalence relations. To keep them separate we give them different names and different symbols. In our geometry we have used the following equivalence relations.

(a) Identity. The relation "is the same as" is easily seen to satisfy the three properties listed above. The word "equal" and the symbol "=" are reserved for this equivalence relation.

(b) Congruence. Here again, the properties are easily checked. (Refer to the talk on Congruence for a general treatment). The symbol is "$\cong$".

(c) Similarity. Here again we have an equivalence relation, denoted by "$\sim$".

(d) We have not introduced any special notation for "equality of area", or "equality of volume", but each of these relations is reflexive, symmetric and transitive. We could, if it were convenient, introduce words and symbols for these equivalence relations.

Such insistence on exactitude of language and symbolism may sometimes seem mere quibbling, but it is on such extreme carefulness that modern mathematics is based.

4. Classification and functions. Equivalence relations are connected closely with another concept which is important in mathematics. This is classification. The connection is as follows.

Suppose we have an equivalence relation $\approx$ defined for a certain set S. We can then classify the elements of S into disjoint classes (i.e. no two classes intersect) S_1, S_2, ..., by putting into the same class all elements which are equivalent to each other. Conversely, suppose that we have a classification of S into disjoint classes. Then we can define an equivalence relation by saying that a is equivalent to b if and only if a and b are in the same class. These two constructions

$$\text{Equivalence} \longrightarrow \text{Classification},$$

$$\text{Classification} \longrightarrow \text{Equivalence},$$

are inverses of each other. If we start with an equivalence, pass to its classification, and then pass from this classification to its induced equivalence, we end up with the same equivalence. Similarly, if we start with a classification, form the induced equivalence, then form its induced classification, we end up with the same classification.

An example may make this clearer. Suppose S is the set of all polygons. Let us define $\approx$ among polygons by saying that $P_1 \approx P_2$ if P_1 and P_2 have the same number of sides. (This relation $\approx$ obviously is reflexive, symmetric and transitive). The induced classification is then into triangles, quadrilaterals, pentagons, hexagons, ..., n-gons,... . If we start with this classification, its induced equivalence is: $P_1 \approx P_2$ if P_1 and P_2 are in the same class, i.e., if they are both n-gons (for the same n). This is the same as the original equivalence.

Notice that in this example, our classification was by means of a unique number attached to each polygon, namely the number of sides. Whenever we have a unique number attached to each object of a set S, we have a numerical function $f(a)$. Thus, every numerical function induces a classification: each class consists precisely of those elements a with the same functional value $f(a)$. As another example let S be the set of angles and let $f(a) = m\angle a$. The corresponding equivalence relation is then our familiar congruence $\cong$, between angles.

On the other hand, not every equivalence relation is easily characterized by a function. If S is the set of triangles it is hard to see how the similarity relation, $\sim$, or the congruence relation, $\cong$, can be associated with a function. As a matter of fact this can be done, but the methods involved are well beyond elementary mathematics, as well as being highly artificial.

The Concept of Congruence.

Congruence is a rich and complex idea with many ramifications in geometry - there really is nothing quite like it in algebra. It applies to figures of all kinds - segments, angles, triangles, circular arcs, polygons, truncated pyramids - in fact to any conceivable figure. It plays an essential role in the theory of geometric measure of length, area and volume - it is intimately related to the important concept of rigid motion.

We will examine carefully the conventional theory of congruence and the related theory of linear measure. This will be contrasted with the theory of congruence adopted in our text. Finally we treat the concept of congruence for general figures and its relation to the idea of rigid motion.

I. The Conventional Theory of Congruence and Linear Measure.

1. Congruence in Terms of Size and Shape. The term congruence immediately calls to mind the famous dictum: Two figures are congruent if they have the same size and the same shape. Certainly this statement emphasizes the basic intuitive or informal idea that if two figures are congruent, one is a "replica" of the other. Also it points up the important property that if we know two figures to be congruent we can infer that they have the same area (or volume) and that they are similar.

But this is not the essential issue. It is: Does our dictum define congruence? Is it really a formal definition of the term congruence in terms of more basic ideas? Clearly the answer is no. For the notions size and shape are more complex than congruence. In order to measure (or define) size (area or volume) we try to find out how many congruent replicas of a basic figure (for example square or cube) "fill out" a given figure. So actually it would be more natural and simple to base the theory of size (and shape) on the idea of congruence rather than the reverse.

2. Congruence in Terms of Rigid Motion. But there are other "definitions" of congruence which we must discuss - consider the famous, "Two figures are congruent if they can be made to coincide by a rigid motion". Let us analyze this. Conceived concretely, say in terms of two paper heart-shaped valentines, it affords an excellent illustration of the intuitive idea of congruence and emphasizes again that one is a "replica" of the other. But this illustration, like most physical situations, does not have the precision required for an abstract mathematical concept. Surely we would have to pick up the first valentine and move it with almost infinite gentleness to prevent bending it slightly when getting it to coincide with the second one. And how could we be certain of perfect coincidence of the two valentines? Wouldn't this require perfect eyesight? It is clear that this "definition" interpreted concretely gives us a physical approximation to the abstract idea of congruence but doesn't define it. Moreover it is not even applicable in many physical situations: you hardly could get two "congruent" billiard balls to coincide by a rigid motion.

Should we then conclude that the idea of rigid motion is essentially physical and can not be mathematicized as an abstract geometrical concept? Definitely not. Mathematicians are ingenious and clever people and it might be a mistake to decide beforehand that they could not construct a precise abstraction from a given physical idea. Most familiar mathematical abstractions had their origin in concrete physical situations - certainly geometry had its origin in practical problems of surveying the heavens and the earth.

Let us table for the present the question of whether we can form an abstract geometrical theory of rigid motions. It would seem that a treatment of congruence based on a logically satisfactory theory of rigid motion could not be elementary and would hardly be suitable for a first course. In any case, without deeper analysis, the second "definition" is not a definition at all and might more properly be considered a statement of a property which rigid motions should have: namely, that any rigid motion transforms a figure into a congruent one.

3. Another Definition. Consider and criticize a third suggested "definition": Two (plane) figures are congruent if a copy of the first made on tracing paper can be made to coincide with the second.

4. Congruence of Segments. Since our three "definitions" do not define congruence we must probe more deeply. Here, as so often in solving problems, the imperialist maxim, "Divide and conquer", is very helpful. Instead of tackling the concept of congruence in its most complex form, that is for arbitrary figures, let us begin by considering a simple special case. A line segment -- or as we shall call it -- a segment is one of the simplest and most important geometric figures. We naturally begin by considering congruence of segments.

Let us recall how this is treated in Euclid or in the conventional high school geometry course. Congruent segments, usually called equal segments, are conceived as "replicas" of each other, in general with different locations in space. Congruent segments may coincide or be identical but they don't have to. If segments $\overline{AB}$ and $\overline{CD}$ are congruent we may interpret this concretely to mean $\overline{AB}$ and $\overline{CD}$ are "caliper equivalent" - that is if a pair of calipers is set so that the ends coincide with A and B then without changing the setting, the ends of the calipers can be made to coincide with C and D.

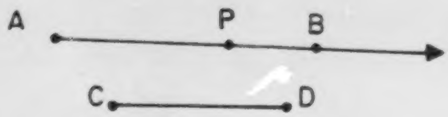
5. *Basic Properties of Congruence of Segments*. What is the *logical significance* of congruence of segments in Euclid? Actually it is taken to be an undefined term. More precisely, using the notation $\overline{AB} \cong \overline{CD}$, congruence is a basic relation $\cong$ between the segments $\overline{AB}$ and $\overline{CD}$ which we do not attempt to define. We study it (as always in mathematics) in terms of its basic properties which are formally stated as postulates. Some of these postulates, which are not explicit in Euclid or in most geometry texts are:

(1) (Reflexive Law) $\overline{AB} \cong \overline{AB}$;

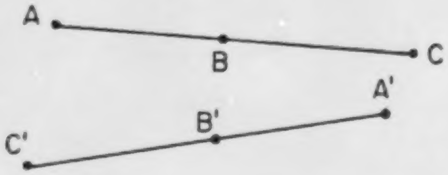
(2) (Symmetry Law) If $\overline{AB} \cong \overline{CD}$ then $\overline{CD} \cong \overline{AB}$;

(3) (Transitive Law) If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$ then $\overline{AB} \cong \overline{EF}$.

That is, congruence of segments satisfies the three basic properties of equality or identity and so is an example of an *equivalence relation*. We must not assume that congruence means identity, since *distinct* segments can be congruent.



(4) (Location Postulate) Let $\overrightarrow{AB}$ be a ray and let $\overline{CD}$ be a segment. Then there exists a unique point P in $\overrightarrow{AB}$ such that $\overline{AP} \cong \overline{CD}$.

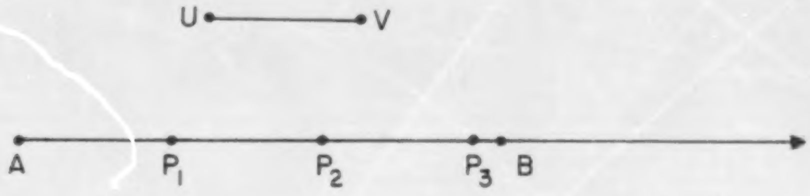


(5) (Additivity Postulate) Suppose $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, B is between A and C and B' is between A' and C'. Then $\overline{AC} \cong \overline{A'C'}$.

We insert a few words on the important mathematical idea of equivalence relation. The most basic example of an equivalence relation and the one which suggests the concept is the relation equality or identity. Identity is distinctive as an equivalence relation in that it satisfies the Principle of Substitution: If b = a we may substitute "b" for "a" in any valid statement involving "a" and obtain a valid statement. Equivalence relations abound in geometry, for example congruence of figures or similarity or equivalence of figures. (For a discussion of equivalence relations see the Talk on Equality, Congruence, and Equivalence.)

6. Theory of Linear Measure. Segments are geometric figures, not numbers. But they can be measured by numbers -- they do have lengths. In the conventional high school treatment it is assumed with little discussion that lengths of segments can be defined as real numbers. We indicate how to do this. Although the result is familiar, the process is complex and subtle and requires for its complete justification additional postulates. However, Postulate (1), ..., (5) above are sufficient for an understanding of the process.

We begin by choosing a segment $\overline{UV}$ which will be unchanged throughout the discussion (a so-called "unit" segment). Now given any segment $\overline{AB}$ we want to measure $\overline{AB}$ in terms of $\overline{UV}$. This involves a "laying-off" process. We take the ray $\overrightarrow{AB}$ and lay off $\overline{UV}$ on it



repeatedly starting at A. Speaking precisely there is a point P_1 in $\overrightarrow{AB}$ such that $\overline{UV} \cong \overline{AP_1}$ (Location Postulate). Similarly we can show that there is a point P_2 in $\overrightarrow{AB}$ such that (a) $\overline{UV} \cong \overline{P_1P_2}$ and (b) P_1 is between A and P_2 which we write (AP_1P_2). The symbol (AP_1P_2) is read P_1 is between A and P_2. Continuing, there is a point P_3 such that $\overline{UV} \cong \overline{P_2P_3}$ and $(P_1P_2P_3)$. By this process we develop a sequence of points $P_1, P_2, \ldots, P_n, \ldots$ on $\overrightarrow{AB}$ such that

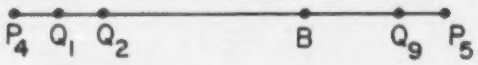
(1) $$\overline{UV} \cong \overline{P_1P_2} \cong \overline{P_2P_3} \cong \ldots \cong \overline{P_{n-1}P_n},$$

(2) $$(AP_1P_2), (P_1P_2P_3), \ldots, (P_{n-2}P_{n-1}P_n).$$

Intuitively (1), (2) say that $\overline{UV}$ is laid off on AB n times in a given direction - but note how very precisely and objectively (1), (2) say this, avoiding the somewhat vague terms "laying off" and "direction". From another viewpoint we are laying the basis for a coordinate system on the line by locating precisely the points $P_1, P_2, \ldots, P_n, \ldots$ which are to correspond to the integers 1, 2,..., n,...

Now what has this to do with the measure of $\overline{AB}$? Clearly we must learn how B is related to the points P_1, P_2, P_3,.... In the simplest case one of these might coincide with B, for example $P_3 = B$. Then of course we define the measure of $\overline{AB}$ to be 3.

7. Refinement of the Approximation Process. You may ask, "Did we have to go through this elaborate process to explain that if the "unit" segment $\overline{UV}$ exactly covers $\overline{AB}$ three times, then the measure of $\overline{AB}$ is 3?" Disregarding the importance of making the idea "exactly covers" mathematically precise, observe that the process helps us to define a measure for $\overline{AB}$ in the more general and difficult case when no one of the points P_1, P_2,... coincides with B. For suppose B falls between two consecutive points of our sequence, say (P_4BP_5). Clearly then we will have to assign to $\overline{AB}$ a measure x such that $4 < x < 5$. In other words we have set up a general process which enables us at least to determine an approximation to the measure of $\overline{AB}$, that is to find lower and upper bounds for it.



We do not complete the discussion but indicate how it proceeds. To fix our ideas, suppose (P_4BP_5). To get a better idea of what the measure of $\overline{AB}$ should be we naturally subdivide $\overline{P_4P_5}$ into ten congruent subsegments and proceed as above. Precisely we set up a subsidiary sequence of points Q_1,.....Q_9 which divide $\overline{P_4P_5}$ into ten congruent subsegments. That is we require

$$\overline{P_4Q_1} \cong \overline{Q_1Q_2} \cong \overline{Q_2Q_3} \cong \ldots \cong \overline{Q_9Q_5}$$

and

$$(P_4Q_1Q_2),(Q_1Q_2Q_3),\ldots, (Q_8Q_9P_5).$$

If B were to coincide with one of $Q_1, Q_2, \ldots, Q_9$ say $B = Q_6$ we assign to $\overline{AB}$ the measure 4.6. If B falls between two of the Q's say (Q_6BQ_7) we require that x, the measure of $\overline{AB}$, satisfy

$$4.6 < x < 4.7.$$

In the latter case we repeat the process by subdividing $\overline{Q_6Q_7}$ into ten congruent subsegments and proceed as before.

8. The Definition of Linear Measure. Clearly we have a complex process (though a refinement of a simple idea) which will assign to segment $\overline{AB}$ a definite decimal, terminating or endless. This decimal we define to be the measure or length of $\overline{AB}$.

9. Basic Properties of Linear Measure. We write the measure of $\overline{AB}$ ($\overline{UV}$ still being fixed) as $m(\overline{AB})$. Observe that we really have here a function $\overline{AB} \longrightarrow m(\overline{AB})$ which associates to each segment a unique positive real number. What are the basic properties of this "measure" function? They are easily grasped intuitively:

(1) $m(\overline{AB}) = m(\overline{A'B'})$ if and only if $\overline{AB} \cong \overline{A'B'}$ - that is, congruent segments and only congruent segments have equal measures;

(2) If (ABC) then $m(\overline{AB}) + m(\overline{BC}) = m(\overline{AC})$ - that is, measure is additive in a natural sense;

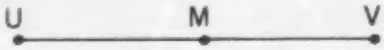
(3) $m(\overline{UV}) = 1$ - that is, the measure of the unit segment is unity.

Notice that (2) is a clear and useful form of the vague statement, "the whole is the sum of its parts".

We summarize in a theorem which can be deduced from a suitable set of postulates for Euclidean Geometry:

Theorem. Let the segment $\overline{UV}$ be given. Then there exists a function which assigns to each segment $\overline{AB}$ a unique positive real number $m(\overline{AB})$ satisfying (1),(2),(3) above.

10. Uniqueness of Measure Function. We naturally ask is there just one measure function? Clearly not. For the function must depend on the choice of the unit segment $\overline{UV}$. To be specific, suppose we take as a new unit segment, $\overline{UM}$, where M is the mid-point of $\overline{UV}$ (that is $\overline{UM} \cong \overline{MV}$ and (UMV)). Then according to our theorem there will be a measure function, let us call it m' (since we have no right to assume it is the same as the original measure function) such that $m'(\overline{UM}) = 1$. We see quickly that $m'(\overline{UV}) = 2$; further it can be shown $m'(\overline{AB}) = 2m(\overline{AB})$ for any segment $\overline{AB}$. This is a formal statement of the trivial seeming fact that "halving the unit of measurement doubles the measure". A corresponding result holds in general:



Theorem. If m,m' are two measure functions on the set of all segments then

$$m'(\overline{AB}) = k \cdot m(\overline{AB})$$

where k is a fixed positive real number.

In the preceding example we had k = 2. Of course k need not be an integer - it can be any positive real number, rational or irrational. As a related example consider the corresponding situation in the measure of angles: The radian measure of an angle is $\frac{\pi}{180}$ times the degree measure of the angle.

Summary: Any two measure functions on the set of all segments are proportional.

What does this mean for the development of the theory of measurement of segments? It says in effect that it doesn't matter which measure function we choose, since making a different choice would only multiply all measures by a constant. Thus, in conventional geometrical theory, we fix a unit $\overline{UV}$ at the beginning, determine a corresponding measure function, and thereafter use this measure function as if it were the only possible one. And instead of saying precisely the measure of $\overline{AB}$ *in terms of unit* $\overline{UV}$, we say simply the measure of $\overline{AB}$, and forget about $\overline{UV}$. The situation in everyday life is quite different - we employ measure functions based on a variety of units: inches, light years, millimeters, miles.

We close this part of our discussion by observing that the distance between A and B is merely defined to be the measure of $\overline{AB}$. Sometimes we want to refer to the distance between A and A itself, this we take to be 0 - a separate definition is required for this case since we may not refer to the segment $\overline{AB}$ unless we know $A \neq B$.

Query. Was it necessary to use the integer ten in the subdivision process? Would others work? Could the process be simplified by making a different choice?

II. *Congruence Based on Distance.*

In this part we discuss the treatment of congruence adopted in the text, contrasting it with the conventional one. The point of departure is to "reverse" the conventional treatment and *define* congruence in terms of distance. This enables us to use our knowledge of the real number system early in the discussion - it leads to a new treatment of the important geometric relation, betweenness, and a new way of conceiving segments and rays.

1. *The Student's Viewpoint.* The conventional treatment, in brief, begins with an undefined notion of congruence of segments

and deduces the existence of a distance function from a suitable set of postulates. The high school student - in studying this treatment - somehow absorbs the idea that segments (and angles) can be measured by numbers, and is permitted to apply his knowledge of algebra whenever it is convenient.

2. *The Distance Postulate*. Since the student *thinks* of segments and angles as measurable by numbers and it is hopeless to *prove* this at his level from non-numerical postulates, it seems most reasonable to make the existence of a measure function or distance a basic postulate which is used consistently throughout the course. So we adopt

Postulate 6. (The Distance Postulate). To every pair of different points P,Q there corresponds a positive real number, called the distance between P and Q and denoted PQ.

Don't read into this more than it says - it is a very weak statement. Notice that it doesn't state a single property of distance - merely that there is such a thing. In particular it doesn't say anything about lengths of segments - in fact we don't even have segments at this stage of our theory.

3. *The Distance Postulate Causes a Change in Viewpoint*. This may seem strange, but it isn't. Referring to Postulates 1,....,5 in the text we see they involve the undefined terms point, line, plane and that they state the basic "incidence" relations telling how points, lines, planes are incident with or "lie on" each other. We have not yet introduced the ideas of *segment* and *ray* which are essential in Euclidean geometry. In Euclid these ideas essentially are taken as undefined. But having adopted the Distance Postulate

we *can* define them. This is an important - and unforeseen - consequence of the Distance Postulate: We don't get just Euclid with the theorems rearranged, but new insights into the basic geometric ideas and a new way of interrelating them.

4. *Between and Segment as Defined Terms*. How then can we *define* segment in terms of the basic terms point, line, plane? It is easy to do this using the additional notion of a point being *between* two points. Having adopted Postulate 6, the idea of distance is at our disposal and we can define betweenness so:

Definition. Let A,B,C be three collinear points. If $AB + BC = AC$ we say B is *between* A and C and we write (ABC).

We now define segment in terms of betweenness.

Definition. Let A,B be two points. Then *segment* $\overline{AB}$ is the set consisting of A and B together with all points that are between A and B. A and B are called *endpoints* of $\overline{AB}$. Further we define $m(\overline{AB})$, the *measure* or *length* of $\overline{AB}$, merely to be the number AB.

That is the length of a segment is merely the number which is the distance between its endpoints. The contrast with conventional theory is striking: There congruence of segments is basic and a difficult argument is needed to prove the existence of a measure function - here distance is basic and the proof of the existence of a measure function is trivial.

5. *Congruence of Segments by Definition*. Now it is absurdly easy to *define* congruence of segments.

Definition. $\overline{AB} \cong \overline{CD}$ means that the lengths of $\overline{AB}$ and $\overline{CD}$ are equal, that is $AB = CD$.

Formally what we have done is just this. We took the basic property relating congruence and measure ((1) p.7 above)

$$m(\overline{AB}) = m(\overline{CD}) \text{ if and only if } \overline{AB} \cong \overline{CD},$$

which is a *theorem* in the conventional treatment, and adopted it as a *definition* in our treatment. There, segments which were congruent were *proved* to have the same measure - here, segments which happen to have the same measure are *called* congruent.

6. *Properties of Congruent Segments.* Does congruence of segments, as we have defined it, have the properties we expect? We see quickly that $\cong$ is an equivalence relation, that is

(1) $\overline{AB} \cong \overline{AB}$;

(2) If $\overline{AB} \cong \overline{CD}$ then $\overline{CD} \cong \overline{AB}$;

(3) If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$ then $\overline{AB} \cong \overline{EF}$.

These merely say

(1') $AB = AB$;

(2') If $AB = CD$ then $CD = AB$;

(3') If $AB = CD$ and $CD = EF$ then $AB = EF$,

which are the basic properties of equality of numbers.

Further we have

(5) Suppose $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, (ABC) and $(A'B'C')$. Then $\overline{AC} \cong \overline{A'C'}$.

To prove this we have

$$AB = A'B',$$
$$BC = B'C',$$

so that

$$AB + BC = A'B' + B'C'.$$

The betweenness relations yield

$$AB + BC = AC, \qquad A'B' + B'C' = A'C',$$

and we get

$$AC = A'C' \quad \text{or} \quad \overline{AC} \cong \overline{A'C'}.$$

Thus several of Euclid's (or Hilbert's) postulates for congruence reduce, in our treatment, to elementary properties of real numbers.

7. The Independence of the Location Property. You may wonder if we can also derive from Postulates 1,...,6 the Location Property ((4) p.4):

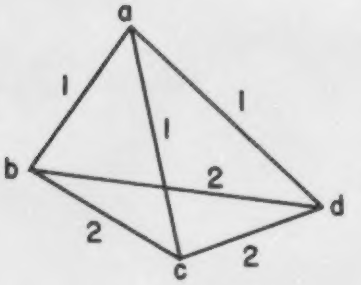
Let $\overrightarrow{AB}$ be a ray and let $\overline{CD}$ be a segment. Then there exists a unique point P in $\overrightarrow{AB}$ such that $\overline{AP} \cong \overline{CD}$.

The answer is - with a vengeance - no. On the basis of Postulates 1,...,6 we can't even prove that a line contains more than two points! To justify this consider the following concrete system. (Compare similar discussion for Postulates 1,...,5 in Commentary for Teachers p.13 starting four lines from the bottom.)

Let the undefined term point be interpreted concretely to be one of the letters a,b,c,d; that is point denotes an element of the set {a,b,c,d}. Line is any couple of these elements; plane is any triple of these elements. Further the undefined term distance is interpreted thus: ab = ac = ad = 1; bc = bd = cd = 2.

Our system described in tabular form:

Point	Line	Plane	Distance
a,b,c,d	{a,b},{a,c},{a,d} {b,c},{b,d},{c,d}	{a,b,c},{a,b,d} {a,c,d},{b,c,d}	ab = ac = ad = 1 bc = bd = cd = 2
4 elements	6 pairs of elements	4 triples of elements	



The diagram gives a compact representation of the system, or it may be thought of as an "equivalent" concrete system in its own right. The representation by means of a tetrahedron is graphical and suggests relations to Euclidean geometry - however like any diagram it may mislead you. For example the "lines" are not the ink streaks which join pairs of dots but just pairs of dots.

It is not difficult to verify that our system satisfies Postulates 1,...,6. For example Postulate 2, considered as an assertion about the system, says:

Any two elements of {a,b,c,d} are contained in a unique couple of elements of {a,b,c,d}. And Postulate 5:

If two triples of elements of {a,b,c,d} have a common element their intersection is a pair of elements.

This is trivial since each two such triples have exactly two common elements. Obviously Postulate 6 is satisfied - in fact we could have chosen the six distances to be any positive real numbers whatsoever.

To signalize that the system satisfies Postulates 1,...,6 we call it a model or interpretation of this set of postulates.

Now we can justify that the Location Property is not deducible as a theorem from Postulates 1,...6. For suppose it were. Then -

since our model satisfies Postulates 1,...,6 - it would have to satisfy the Location Property. Consequently any segment could be "laid off" on line ab as a congruent segment. Hence each number which is a distance between two points of our model could be realized as a distance between two points of line ab. This is absurd, since the only distance between two points of line ab is 1 and the distance cd is 2. Thus the Location Property is not deducible from Postulates 1,...,6 or as we say is _independent_ of these postulates.

8. _The Ruler Postulate_. Clearly Postulates 1,...,6 are too weak to support the kind of theoretical structure we are trying to build. The text supplements them by adopting the powerful Ruler Postulate:

Postulate 7. (The Ruler Postulate). The points of a line can be placed in correspondence with the real numbers in such a way that

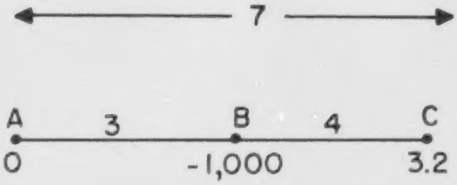
(1) To every point of the line there corresponds exactly one real number,

(2) To every real number there corresponds exactly one point of the line, and

(3) The distance between two points is the absolute value of the difference of the corresponding numbers.

This guarantees at one swoop that a line has the intrinsic properties we expect of it. Now the lines in every model of our theory will be well-behaved and richly endowed with points. It implies the congruence _and_ order properties of a line in the conventional theory. Specifically it yields: (1) a form of the Location Property (Theorem 4-3); (2) that any segment can be "extended by any given length" (Theorem 4-4); (3) that a segment can be "divided" into a given number of congruent "parts" - in particular it can be bisected (Theorem 4-5). It implies important order properties: Theorem 4-1 which says in effect that the order of points on a line in terms of geometric betweenness corresponds exactly to the order of their coordinates in terms of algebraic betweenness; and the Line-Separation Property which is not explicitly dealt with in the text (see Commentary For Teachers p.23).

Observe the attractive interdependence of the weak Distance Postulate and the powerful Ruler Postulate. The first asserts the existence of a distance function but permits it to be as trivial as the one in the finite model above - the second tailors the line to our expectations but is impossible of statement without the notion distance postulated in the first. Note that if we weaken the Ruler Postulate by dropping condition (3) and require merely the existence of a 1-1 correspondence between the points of a line and the set of real numbers, we may have pathological situations of the type indicated in the diagram. Here B is between A and C since AB + BC = AC - but - 1,000, the coordinate of B, definitely is not between the coordinates of A and C.



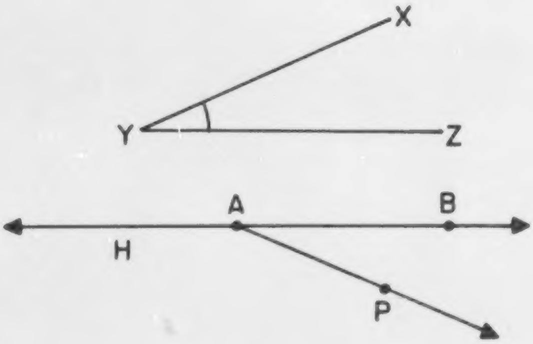
Our discussion suggests an important point in mathematical or deductive thinking. The Distance Postulate enables us to define betweenness but not to prove the existence of a single point between two given points. This is illustrated by the finite model above. The Ruler Postulate however implies the existence of infinitely many points between any two. This illustrates the point that a mathematical definition does not assert the existence of the entity defined. You may characterize the pot of gold at the end of the rainbow with great precision but you may experience equally great disappointment if you start to search for it before proving an existence theorem.

A final word. We may have oversold the deductive power of the Ruler Postulate and given you the impression that Postulates 1,...,7 are sufficient for a complete theory of congruence. This is not so. Our theory so far is sufficient for the "linear" theory of congruence, specifically for congruence of segments - but not for congruence of more general figures like angles, triangles, circular arcs or triangular pyramids. For this we must introduce further postulates concerning congruence of angles and triangles. We discuss this in the next part since our main object here has been to indicate the flavor of the treatment in the text in contrast with the conventional one.

III. Congruence for Arbitrary Figures and Rigid Motions.

In this part we continue the discussion of congruence by indicating how it is successively defined for familiar elementary figures: angles, triangles, etc. Then using the simple and powerful modern idea of transformation we formulate the congruence concept for arbitrary figures - this surpasses in elegance and generality anything obtained in the field by the classical geometers. As a byproduct we obtain - after two millenia - a precise mathematical concept of rigid motion. This is a great cultural achievement of our time. Rescuing from the jungles of physical intuition Euclid's crude superposition argument, we refine and perfect it to yield an objectively formulated concept which will be of use to human beings as long as they are impelled to think precisely about space.

1. Congruence of Angles. The conventional treatment of angle congruence is similar to that sketched in Part I for congruence of segments - but naturally is a bit more complicated since angles are more complex figures than segments. It begins with an undefined relation $\angle ABC \cong \angle PQR$ between two angles which as usual indicates that they are replicas of each other. This may be interpreted concretely to mean that if a frame composed of two jointed rods is set so that the rods coincide with the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, then without changing the setting the rods can be made to coincide with $\overrightarrow{QP}$ and $\overrightarrow{QR}$. We assume as for segments that congruence of angles is an equivalence relation:



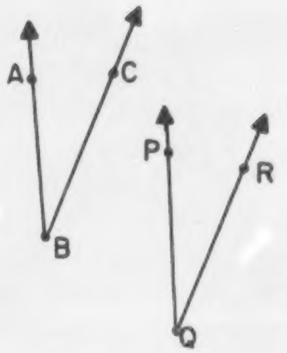
(1) (Reflexive Law) $\angle ABC \cong \angle ABC$;

(2) (Symmetry Law) If $\angle ABC \cong \angle PQR$ then $\angle PQR \cong \angle ABC$;

(3) (Transitive Law) If $\angle ABC \cong \angle PQR$ and $\angle PQR \cong \angle XYZ$ then $\angle ABC \cong \angle XYZ$.

The Location Postulate for segments (see (4) p.4) has the analogue

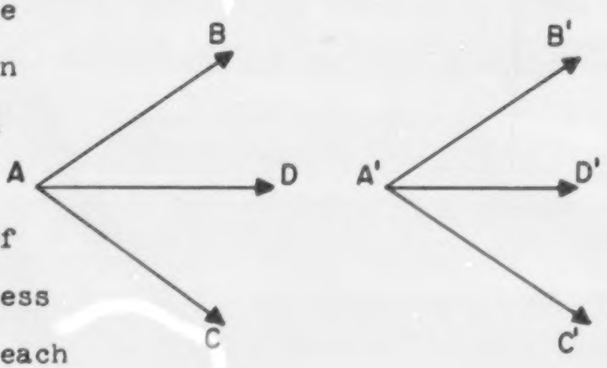
(4) (Angle Location Postulate)



Let $\angle XYZ$ be any angle and $\overrightarrow{AB}$ be a ray on the edge of half-plane H. Then there is exactly one ray $\overrightarrow{AP}$, with P in H, such that $\angle PAB \cong \angle XYZ$.

And the Additivity Postulate ((5) p.4) appears in the form

(5) (Angle-Additivity Postulate) Suppose $\angle BAD \cong \angle B'A'D'$, $\angle DAC \cong \angle D'A'C'$, D is in the interior of $\angle BAC$ and D' is in the interior of $\angle B'A'C'$. Then $\angle BAC \cong \angle B'A'C'$.



Essentially on the basis of these postulates a measure process can be set up which assigns to each angle a unique positive real number called its measure in such a way that a fixed preassigned angle ("unit" angle) has measure 1 (compare Part I 6,7,8,19).

Denoting the measure of $\angle XYZ$ by $m \angle XYZ$, we have as you would expect from our discussion of measure of segments:

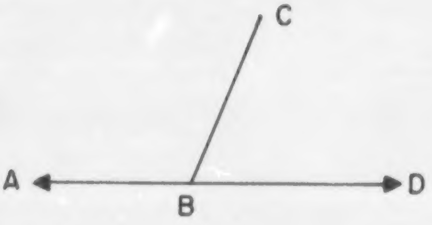
(1) $m \angle ABC = m \angle A'B'C'$ if and only if $\angle ABC \cong \angle A'B'C'$;

(2) If C is interior to $\angle ABD$ then

$m \angle ABC + m \angle CBD = m \angle ABD$.

(Compare (1),(2) p.7.)

But there are two properties which are unique to angular measure. First there is a real number b which is a least upper bound for the measure S of all angles (b is 180 in the familiar "degree measure"). Second the measure S of "supplementary adjacent" angles always have a constant sum and this sum is the least upper bound b. Stated precisely: If $\angle ABC$ and $\angle CBD$ are such that $\overrightarrow{BA}$ and $\overrightarrow{BD}$ are opposite rays then $m \angle ABC + m \angle CBD = b$.



2. Congruence of Angles Based on Angular Measure. We saw in (1) above that the conventional theory of angle congruence yields (as for segments) that two angles are congruent if and only if they have equal measures. This suggests (as for segments) that we assume the existence of angular measure and define congruence of angles in terms of it. Thus the treatment in the text assumes

Postulate 11. (The Angle Measurement Postulate) To every angle $\angle ABC$ there corresponds a real number between 0 and 180, called the measure of the angle, and written as $m \angle ABC$, (compare the Distance Postulate).

Clearly our postulate has been set up so that the unit angle is the degree. In other words the angle characterized by $m \angle ABC = 1$ is what is usually defined to be a degree and will have the property that ninety such angles laid "side by side" will form a right angle. Precisely speaking the measure of a right angle will turn out to be 90. Notice that the measure of no angle can be 0 or 180 since our definition of angle restricts the side S to be non-collinear. (For a discussion of this restriction see Commentary For Teachers p.29 Comment 106.)

Now following a familiar path (Part II § 5) we adopt the

Definition. $\angle ABC \cong \angle PQR$ means that $m \angle ABC = m \angle PQR$. Then properties (1),(2),(3) of § 1 above reduce to familiar equality properties of real numbers. The Angle Location Property ((4) of §1) must be postulated and is introduced in the form:

Postulate 12. (The Angle Construction Postulate) Let $\overrightarrow{AB}$ be a ray on the edge of half-plane H. For every number r between 0 and 180 there is exactly one ray $\overrightarrow{AP}$, with P in H, such that $m \angle PAB = r$.

It might be thought now that the additivity property for angles ((5) of § 1) could be derived as a theorem as was the corresponding property for segments (see (5) p.12). This isn't so. But it is a simple and important property of angles, and it is perfectly natural to postulate it:

Postulate 13. (The Angle-Addition Postulate) If D is a point in the interior of $\angle BAC$, then $m \angle BAC = m \angle BAD + m \angle DAC$.

Finally we need a postulate to express the peculiarly "angular" property of supplementation:

Postulate 14. (The Supplement Postulate) If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays and $\overrightarrow{AD}$ is another ray, then $m\angle DAC + m\angle DAB = 180$.

3. Congruence of Triangles. We are now ready to consider congruence of triangles. Our definition of congruent triangles (Chapter 6 of text) is essentially the conventional one: One triangle is a "copy" of the other in the sense that its parts are "copies" of the corresponding parts of the other. But observe the precision with which it is formulated. The correspondence doesn't depend on individual interpretation of the vague term "corresponding" but is based objectively on a pairing of the vertices

$$A \longleftrightarrow A', \quad B \longleftrightarrow B', \quad C \longleftrightarrow C'$$

which induces a pairing of sides and of angles

$$\overline{AB} \longleftrightarrow \overline{A'B'}, \quad \overline{BC} \longleftrightarrow \overline{B'C'}, \quad \overline{CA} \longleftrightarrow \overline{C'A'}$$

$$\angle A \longleftrightarrow \angle A', \quad \angle B \longleftrightarrow \angle B', \quad \angle C \longleftrightarrow \angle C'.$$

Notice how spelling out the notion "corresponding" in this way helps to point up the importance of the notion of a congruence which is not mentioned in the conventional treatment. Thus our treatment brings to the fore the idea of a 1-1 correspondence between the vertices of ΔABC and $\Delta A'B'C'$ which ensures that they are congruent because it requires corresponding sides and corresponding angles to be congruent, that is to have equal measures. This simple idea is capable of broad generalization.

Do we need postulates on congruence of triangles? We have a lot of information on congruence of segments and congruence of angles, separately - but nothing to interrelate these ideas. For example we can't yet prove the base angles of an isosceles triangle are congruent. Thus we introduce the S.A.S. Postulate to bind together our knowledge of segment congruence and angle congruence.

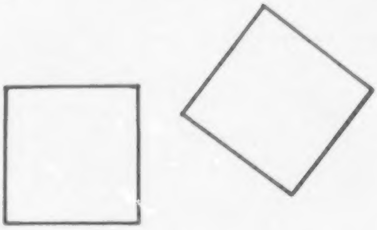
Now let us examine more closely the notion of congruence of triangles. Is it really necessary to require "concurrence" of six parts of each of the triangles? If we think of the sides of a triangle as its basic determining parts it seems very natural to define congruent triangles as having corresponding sides which are congruent. Naturally if we were to adopt this definition we would assume S.S.S. as a postulate in order to ensure that our definition of congruent triangles is equivalent to the familiar one. Notice how much simpler the definition of a congruence between triangles becomes if we adopt the suggested definition. It is merely a 1-1 correspondence between the vertices of the triangles,

$$A \longleftrightarrow A', \quad B \longleftrightarrow B', \quad C \longleftrightarrow C'$$

which "preserves" distances in the sense that the distance between any two vertices of one triangle equals the distance between their corresponding vertices in the second triangle, that is

$$AB = A'B', \quad BC = B'C', \quad AC = A'C'.$$

4. Congruence of Quadrilaterals. The main objection to the suggested definition is that it doesn't generalize in the obvious way for polygons - not even for quadrilaterals. This attested by the fact that a square and a rhombus can have sides of the same length and not be congruent. So to guarantee congruence of quadrilaterals it is not sufficient to require just that corresponding sides be congruent, and it is customary to supplement this by requiring the congruence of corresponding angles. Thus the conventional definition requiring congruence both of sides and of angles applies equally well to triangles and quadrilaterals.



However angles though very important, are rather strange creatures compared to segments and it seems desirable, if possible, to characterize congruent quadrilaterals in terms of congruent segments, or equivalently, equal distances. This is not so hard.

Going back to a triangle we observe that its three vertices taken two at a time yield three segments or three distances and that the figure is in a sense determined by these three distances. Similarly the four vertices of a quadrilateral yield not four but six segments (the sides and the diagonals) and six corresponding distances, which serve to determine the quadrilateral. This suggests: If we have a 1-1 correspondence

$$A \longleftrightarrow A', \quad B \longleftrightarrow B', \quad C \longleftrightarrow C', \quad D \longleftrightarrow D'$$

between the vertices of the quadrilaterals ABCD, A'B'C'D' such that corresponding distances are preserved, that is

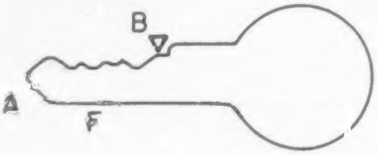
$$AB, AC, AD, BC, BD, CD = A'B', A'C', A'D', B'C', B'D', C'D'$$

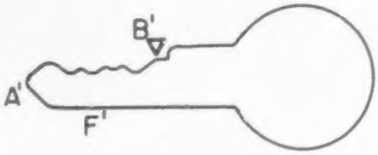
we call the correspondence a congruence and we write $ABCD \cong A'B'C'D'$. It is not hard to show this definition equivalent to the more familiar one.

5. Congruence of Arbitrary Figures. We now must face the problem of formulating a general definition of congruence. The piecemeal process we have employed, defining congruence separately for segments, angles, triangles, quadrilaterals is unavoidable in an elementary treatment but is neither satisfying nor complete. For it still remains to define congruent circles and congruent circular arcs and congruent ellipses and congruent rectangular solids, etc. In each case we construct an appropriate definition, we are sure it is correct and are equally sure the general concept has eluded us.

So let's make a fresh start. Suppose F and F' are two congruent figures. Our basic intuition is that F' is an exact copy of F. Somehow this entails that each "part" of F' copies a corresponding "part" of F - that each point of F' behaves like some corresponding point of F. If F has a sharp point at A then F' must have a sharp point at a corresponding point A'; if F has maximum flatness at B then F' has maximum flatness at a corresponding point B'; if F has a largest chord $\overline{PQ}$ of length 12.3 then F' has a corresponding largest chord $\overline{P'Q'}$ of the same length - 12.3; and so on. How can we tie together these illustrations in a simple and precise way?

6. *A Congruence Machine*. Suppose instead of conceiving F' as a given copy of F, we take F and try to make a copy F' of it. As an illustration let F be a house key. Then F' can be produced by a key duplicating machine. The machine has the secret of the congruence concept - how does it work?





The machine has two moving parts: a scanning bar which traces the given key and a cutting bar which cuts a blank into a duplicate. As the scanning bar traces F starting at its tip A, the cutting bar traces the blank starting at its corresponding tip A'. As the scanner moves to position B, the cutter cuts away the metal and comes to rest at a corresponding position B'. When B rises to a "peak" so does B' - when B falls to a trough so does B' - when B traverses a line segment, B' traverses a line segment of equal length.

What guarantees that this process yields a true copy? Simply this: When the scanner is fixed in a position B, the cutter comes to rest in a position B' such that distances AB and A'B' are equal. *And this is true for each position B of the scanner*. Clearly what the machine does is to associate to each chord $\overline{AB}$ from A of F an "equal" chord $\overline{A'B'}$ from A' of F'. And it associates the chords by associating their endpoints B and B'. Precisely speaking, the machine effects a 1-1 correspondence $X \longleftrightarrow X'$

between F and F' such that the distance AX always equals the distance A'X'.

Does this property hold just for A, the tip of F, and A' its correspondent in F'? Clearly not. The machine doesn't know where we start. What we have asserted about the chords of F from A will hold just as well for the chords from any point of F. So the 1-1 correspondence $X \longleftrightarrow X'$ between F and F' has the stronger property that for every choice of P and Q if $P \longleftrightarrow P'$, $Q \longleftrightarrow Q'$ then $PQ = P'Q'$ - or as we say the correspondence preserves distance. Here we have the essence of the concept of congruence.

The legend has it that when Pythagoras succeeded in proving the theorem ascribed to him he was so elated that he sacrificed a hecatomb of oxen to the gods. Surely in the light of this tradition the formal definition of congruence deserves a section all to itself.

7. The Definition. Let $X \longleftrightarrow X'$ be a 1-1 correspondence between two sets of points F, F' such that

$$P \longleftrightarrow P', \quad Q \longleftrightarrow Q'$$

always implies $PQ = P'Q'$. Then we say F is congruent to F' and we write $F \cong F'$. Moreover we call the 1-1 correspondence a congruence between F and F'.

This definition is the culmination of two thousand years of thinking about congruence. Although it may seem quite abstract it unifies and unites the piecemeal discussion of congruence we have given. Every instance of congruent figures discussed above from segments to quadrilaterals can be proved to be a case of our general definition. This is discussed in detail in Appendix VI of the text on Rigid Motion.

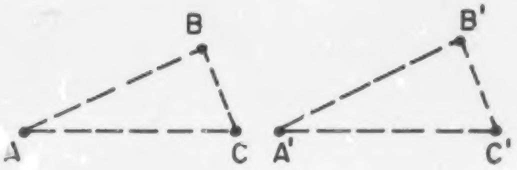
As a simple illustration of the definition let F and F' each be a triple of non-collinear points, say F is $\{A,B,C\}$ and F' is $\{A',B',C'\}$. Let the 1-1 correspondence between F and F' which preserves distance be

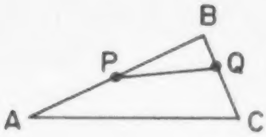
(1) $A \longleftrightarrow A'$, $B \longleftrightarrow B'$, $C \longleftrightarrow C'$.

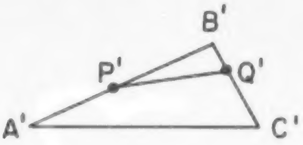
Then we have

$AB = A'B'$, $BC = B'C'$, $AC = A'C'$.

We see intuitively that F' is a copy of F. Now shift from the point triples to the triangles they determine. The S.S.S. Theorem tells that $\triangle ABC$ is congruent to $\triangle A'B'C'$ in the conventional sense.



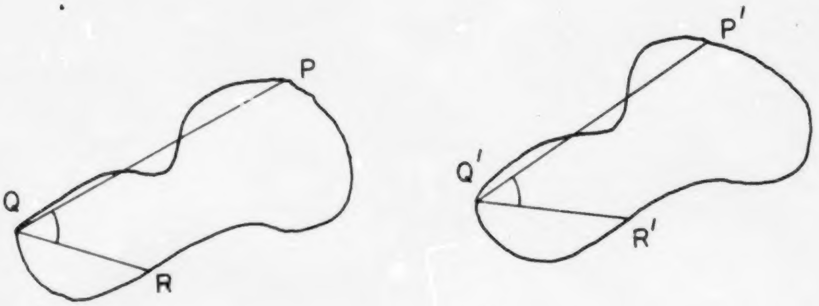




It follows (see Appendix VI of text) that $\triangle ABC \cong \triangle A'B'C'$ in the sense of our definition. Actually there is a 1-1 correspondence between the infinite point sets $\triangle ABC$, $\triangle A'B'C'$ which makes the vertices correspond as in (1) and which has the property that $P \longleftrightarrow P'$, $Q \longleftrightarrow Q'$ always implies $PQ = P'Q'$.

Observe how the correspondence between the triangle is engendered by the trivial seeming correspondence between their vertices. For example if P is on $\overline{AB}$ its correspondent P' is determined as the unique point P' on $\overline{A'B'}$ such that $A'P' = AP$. Let us think of the finite set of its vertices, $\{A,B,C\}$, as a "skeleton" of $\triangle ABC$. Then if the skeletons $\{A,B,C\}$, $\{A',B',C'\}$ of two triangles are congruent the triangles as a whole are congruent - using "congruent" in its present sense. This idea was too complex to introduce in Chapter 6 of the text. But it was foreshadowed there in the insistence that congruence of triangles was the consequence of the existence of a "congruence" between them - that is a 1-1 correspondence between their sets of vertices which preserves lengths of sides and measures of angles.

There is an essential element of complexity in the definition of congruence: It requires (in general) the pairing off of the points of two infinite sets so as to preserve distance. This is unavoidable - it even seems to be present in the comparatively simple problem of duplicating keys. There is however an important element of simplicity: We don't have to mention angles and the preservation of their measures - the distance concept covers the situation. It follows easily that angle measures are preserved:



for if $P \longleftrightarrow P'$, $Q \longleftrightarrow Q'$, $R \longleftrightarrow R'$ correspond under a congruence between F and F', and P,Q,R are non-collinear, we see by the S.S.S. Theorem that $m \angle PQR = m \angle P'Q'R'$.

You may find it interesting to give for quadrilaterals a discussion like the above for triangles - consider the vertex sets {A,B,C,D}, {A',B',C',D'} of quadrilaterals ABCD, A'B'C'D' as their "skeletons". In this connection recall the discussion of congruence of quadrilaterals at the end of § 4.

8. Motion in Geometry. We can state the definition of rigid motion now, but it probably will be more meaningful if we say a few words first about the sense in which "motion" is used in contemporary geometry.

Let a body B move physically from an initial position F in space to a final position F'. It is not necessary for our purposes in geometry (as compared say with kinematics or fluid dynamics) to bother about the intermediate stages of the motion. So we can describe the motion merely by specifying the initial position X in F of an arbitrary point P of body B and its corresponding final position X' in F'. In its most general form, then, a motion is conceived as a 1-1 correspondence or transformation between two figures F and F'. The technical term "transformation" is often preferable to "motion" since it doesn't suggest various irrelevant attributes of physical motion.

9. Rigid Motion. A motion or transformation between two point sets F and F' is a rigid motion if it preserves distances - that is if it is a congruence between F and F' as defined in § 7. A detailed discussion of the concept of rigid motion appears in Appendix 6 of the text.

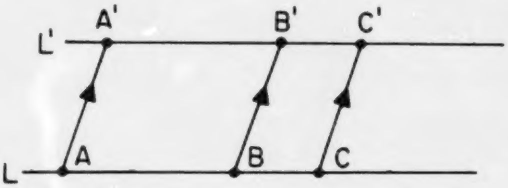
To introduce you to the modern theory of congruent figures and rigid motion we have put the main emphasis on the first, since it is more familiar and seems easier to apprehend. However glancing back at the definition of congruent figures, you see it implicitly involves the notion of rigid motion. In fact now we can reword it: F is congruent to F' provided there exists a rigid motion between them, or as we say more graphically, a rigid motion which "transforms F into F'". This is the highly refined culmination of the vague and famous classical statement which served to introduce our discussion of congruence: "Two figures are congruent if they can be made to coincide by a rigid motion".

Sometimes the clarification of the basic concepts of a branch of mathematics firms up the foundations, puts the capstone on the superstructure and sets it to rest. This is not so here. The concept of rigid motion has stimulated the study of classical geometry, has yielded new insights and helped to unfold new unities. It has suggested the study of more general geometric transformations ("non-rigid motions") and has presented problems to the field of Modern Algebra, since motions tend to occur in certain "natural algebraic formations" called groups.

In the first place congruence and rigid motion have an impact on geometry since they apply to all figures. We can talk precisely not merely about congruence of (or rigid motion between) triangular pyramids or spherical zones or hyperbolic paraboloids but also of lines, planes, space, half-planes, rays, etc. At first it may sound silly to say a line is congruent to a line - but try to find a better replica of a line than a line! It must be just because the relation congruence applied to lines is so fundamental and universal that we are not conscious of it - as a fish must be unconscious of the notion humidity. In a first approach, congruence takes on importance as applied to segments (or angles or triangles) precisely because not all segments (or angles or triangles) are congruent to each other.

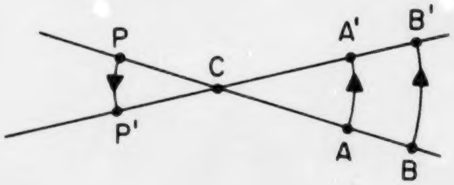
So it may seem trivial to say a line is congruent to a line or a plane to a plane or space to itself. But suppose we shift the focus from the static idea of congruent figures to the dynamic - and logically prior - idea of rigid motion. Is it trivial to say there exist rigid motions between lines or planes or between space and itself? Just to ask this question discloses a broad vista: One of the principal concerns of contemporary geometry (or contemporary mathematics) is the study of transformations (rigid and non-rigid) of n-dimensional spaces.

Consider the simplest case: rigid motions which transform a line L into a line L'. If L || L' we have slides or translations which "move" the points of L along parallel transversals to get their corresponding points of L'. If L and L' meet in just one point C we have a rotation about C. If L and L' coincide, that is L = L', we have two types of rigid motions operating on L:

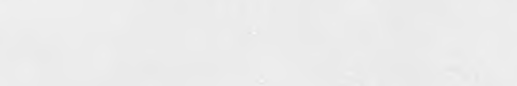


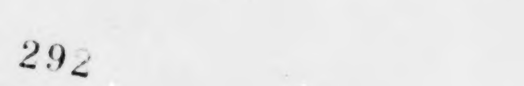
(1) translations along L;

(2) reflections of L in a point C, where point C of L is "fixed" (that is it corresponds to itself) and every other point of L "moves" on L from one side of C to the other.



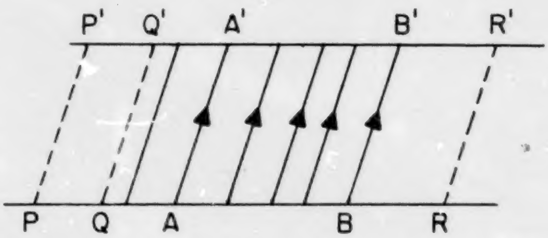
Similar considerations apply to planes. The theory culminates in the study of rigid motions of space - that is between space and





itself. Here the basic types are *translations*, in which no point is fixed, *rotations* in which each point of a line (the axis of the rotation) is fixed, and *reflections* in a plane E in which each point of plane E is fixed and the half-spaces of E are "interchanged". More precisely a reflection in E is a transformation $X \longleftrightarrow X'$ such that if X is in E then $X' = X$ and if X is not in E then E is the perpendicular bisector of $\overline{XX'}$. All rigid motions of space are "combinations" of these three basic types, just as all positive integers other than 1 are combinations of primes.

You may say that the theory of rigid motions of lines, planes and space is attractive and relatively simple, but haven't we left out the annoying complexities involved in the study of specific congruent figures like segments, truncated triangular pyramids and cones with oval bases? Not at all! They are elegantly covered in the theory of rigid motions of the basic "linear manifolds": line, plane, space.



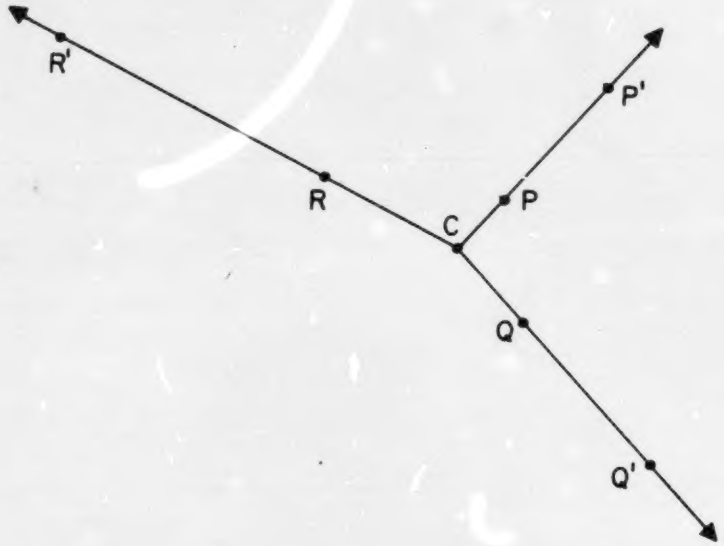
As a very simple illustration suppose segment $\overline{AB}$ is congruent to segment $\overline{A'B'}$. Then there is a rigid motion between them which, let us say, makes A correspond to A' and B to B'. Now we have the remarkable result that this rigid motion, which is a certain kind of 1-1 correspondence between segments $\overline{AB}$ and $\overline{A'B'}$ can be extended to form a rigid motion between the whole line $\overleftrightarrow{AB}$ and the whole line $\overleftrightarrow{A'B'}$ - and this extension can be made in just one way.

Thus we don't disturb the correspondence between $\overline{AB}$ and $\overline{A'B'}$ but "amplify" it by suitably defining a unique correspondent for each point of $\overleftrightarrow{AB}$ not in $\overline{AB}$, so that the final correspondence is a rigid motion between $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$. So in the study of rigid motions between lines as wholes, we are automatically covering all possible rigid motions (and hence all possible relations of congruence) between "linear" figures, that is subsets of lines. Similarly any rigid motion between "planar" figures is uniquely extendable to a rigid motion of their containing planes. Finally we observe that any conceivable rigid motion is encompassed by a rigid motion of space.

10. Non-rigid Motions. As we have indicated modern geometry is concerned with transformations that do not preserve distance as well as those which do. In Euclidean geometry the most important example is a similarity, which bears the same relation to similar figures that a congruence or rigid motion does to figures which are congruent. Formally suppose $X \longleftrightarrow X'$ is a 1-1 correspondence between figures F and F' such that

$$P \longleftrightarrow P', \quad Q \longleftrightarrow Q'$$

always implies $P'Q' = k \cdot PQ$ where k is a fixed positive number. Then we call the correspondence a similarity transformation or a similarity and we say F is similar to F'. It easily follows that a similarity transformation - though not in general a rigid motion - always preserves angle measures. This definition of similar figures, when restricted to triangles, can be proved equivalent to the familiar one. The simplest general type of similarity is the dilatation (in a plane or in space) - this is a similarity which leaves a given point C fixed and radially "stretches" the distance of any point to C by a positive factor k.



Other important types of transformations are central in various geometric theories. For example "parallel projection" between planes in affine geometry; "central projection" between planes in projective geometry; and topological transformations, which are a type of continuous 1-1 correspondence, in topology. The theory of map-making is concerned with various "projections" or other kinds of transformations between a sphere and a cone, cylinder or plane.

And so we have ended our talk by touching upon a modern generalization of rigid motion which well might merit a talk for itself.

Area

It is possible to develop the theory of area, as far as we need it, from a very simple set of postulates, which are intuitively acceptable. In some respects they are more intuitive than the ones given in the text, being simpler to state and requiring fewer preliminary definitions. For example, it is not necessary to define polygonal region in order to state the postulates. It is satisfying that this is one of the many cases in mathematics in which intuition and rigor go hand in hand. We shall sketch this development at least up to the point where it is clear that we could proceed as in the text, by deriving as theorems the postulates of the text which are not already included in our set. Some of the early theorems may appear obvious and hardly worth proving; but if we recognize the fact that postulate systems are constructed by fallible humans and need to be tested by their consequences, then we should derive satisfaction from the provability of some "obvious" statements by means of our postulate system.

We always speak of the area of something, and this something is a region or a figure -- which are simply names for certain sets of points in a plane. Thus, area is a function of sets, an assignment of a unique real number to a set. Whenever we speak of a function, it is important to be quite clear as to the domain of the function, that is, the set of objects for which the function provides us with an answer. In our case, we must ask, what sets are to have an area assigned to them? We could limit ourselves, if we wished, to simple sets, like polygonal regions. This has the disadvantage that it eliminates regions bounded by circles, ellipses, hyperbolas, and other smooth curves, regions which (our intuition tells us) should have areas. Of course, we do not want huge sets like the whole plane, or half-planes, or the interiors of angles, to have area. These all have the property of being unbounded. Fortunately, it can be proved that it is possible to assign a reasonable area to every reasonable set in the plane. The first "reasonable" means that the area function will not violate our intuition. The second "reasonable" we shall interpret in the widest possible sense, namely, as "bounded". A bounded set is one that can be enclosed in some square (or circle). We shall therefore adopt as our first postulate the following:

Postulate 1. There is a function A (called area) defined for all bounded sets in the plane; to each bounded set S, A assigns a unique non-negative number A(S).

Let us observe immediately that a point and a segment are bounded sets, so we have committed ourselves to the unfamiliar position of attributing an area to such sets. The area will turn out to be zero, of course. There are excellent precedents: let us recall that we have allowed ourselves to speak of the distance from a point to itself as being zero. Analogously, in the theory of probability it is useful to have events with zero probability, even though the events are possible. Indeed, the theories of linear measure, area, volume, probability, and counting all have a great deal in common, since they are concerned with assigning measures to various sets. Far from being a disadvantage, the concept of zero area is extremely valuable. It makes explicit our sound intuition of what sets are "negligible" as far as area is concerned.

For example, the area-addition postulate in the text (Postulate 19) essentially asserts that the area of the union of two sets is equal to the sum of their areas, provided that they overlap in a "negligible" set -- a finite union of points and segments. It is somewhat easier to accept an area-addition postulate in which the "negligible" set is the empty set, as in Postulate 2 that follows, and to prove later that certain sets really are "negligible".

Postulate 2. If S and T are bounded sets in the plane which have no points in common, then the area of the union of S and T is equal to the sum of the areas. That is, if V is the union of S and T, then $A(V) = A(S) + A(T)$.

We have already remarked that Postulate 2 is weaker in one respect than the area-addition postulate in the text, for it does not allow even one point in common to the sets S and T. Observe also that Postulate 2 does not need to assert the existence of $A(V)$. This is in fact a simple consequence of Postulate 1, for the union of two bounded sets is also bounded.

Our third postulate will give the essential connection between our geometry and area. For this we need a somewhat more general concept of congruence than the usual one. Two sets will be called congruent if there is a one-to-one correspondence between them which preserves all distances. More precisely, suppose there is a one-to-one correspondence between S and T such that, A and B being any points of S corresponding to A' and B' in T, the distance AB is equal to the distance A'B'. Then we shall say that S is congruent to T, or $S \cong T$. Our definitions of congruence for segments, angles, triangles, and circles are special cases of this more general definition. For a fuller treatment, see the Appendix on Rigid Motion and the Talk on Congruence. If our area function is to be reasonable, then congruent sets should have the same area:

Postulate 3. If S is a bounded set and $S \cong T$, then $A(S) = A(T)$.

Again, it is easy to see intuitively that if S is bounded and $S \cong T$, then T is bounded, and $A(T)$ exists by Postulate 1.

Now let us consider the area of a square of side 1 together with its interior. For all we know from the first three postulates, this area might be 0. This does violence to our intuition, and even more, we could then prove that every bounded set has area 0. Therefore we must postulate that this area is positive, say equal to k. But then the new area function defined by $A'(S) = \frac{1}{k}A(S)$ would be just as good as the old and would have the desirable property that it assigns the value 1 to the unit square and its interior. We shall therefore postulate this immediately:

Postulate 4. If S is the set consisting of a square of side 1 together with its interior, then $A(S) = 1$.

This postulate essentially does no more than (a) rule out the trivial case of a constantly 0 area function, and (b) fix the unit by which we measure the area of a set. We can think of it as a normalization postulate, and shall speak of our area function as being normalized.

Summing up our four postulates -- these are all we need -- we see that we have a non-negative (Post. 1), finitely-additive (Post. 2), normalized (Post. 4) function of bounded sets in the plane (Post. 1), invariant under rigid motion (or congruence) (Post. 3). The term "finitely-additive" refers to the fact that we can easily replace the two sets in Postulate 2 by any finite number of sets, no two of which have a point in common.

On page 1 we stated that it is possible to assign a reasonable area to every reasonable set in the plane. This theorem, asserting the existence of such a function, is rather deep and difficult to prove. Nevertheless, it provides us with a sound basis for a treatment of area in the plane. The set of four postulates matches our intuition quite well, especially if we have not subjected to close scrutiny the vast generality involved in the phrase "all bounded sets in the plane". It should be remarked that the theorem does not guarantee a unique function, but any two functions that satisfy the conditions will agree for decent, non-pathological sets such as polygonal regions, circular regions, and regions bounded by arcs of smooth curves like parabolas, hyperbolas, ellipses, etc.

It would be pleasant if this treatment could be generalized to volume in three dimensions. Surprisingly, the corresponding statement in three dimensions is false. One form of the Banach-Tarski paradox asserts that it is possible to split each of two spheres of different radii into the same finite number of sets, corresponding sets from each sphere being congruent. If the three-dimensional statement were true, the corresponding sets would have equal volumes, by the invariance under congruence, and therefore the spheres would have equal volume, by the finite-additivity of volume. On the other hand, the usual formula for the volume of a sphere would be valid, thus leading to a contradiction. In three dimensions, therefore, it is necessary to limit our volume function to a more restricted class of sets than the bounded ones. This restriction is no cause for alarm, since the resulting domain of the volume function is still much wider than we need for ordinary purposes. The sets that we exclude are all really "wild". With this one modification the methods used here are still applicable in three dimensions.

Now we shall proceed with the business of developing the consequences of our set of postulates. These consequences we shall state as theorems. First, however, we need a simple result which has nothing directly to do with area, but which is a basic property of our real number system.

Theorem 1. If a is a non-negative number such that for every positive integer number n, $na \leq 1$, then $a = 0$.

The statement may seem a little strange, but it is specifically designed to yield the type of result needed, namely that a certain number is 0. For example, suppose that we wish to prove that a certain formula yields the correct value for the area of a given figure. Let the area be A and the number given by the formula be B. Denote by a the absolute value of their difference, $|A - B|$. Then we wish to prove that $a = 0$. We may be able to show that no multiple of a exceeds 1. If so, then Theorem 1 assures us that $a = 0$ and therefore that $A = B$. Another way of stating Theorem 1 is: There is no positive number which is simultaneously $\leq 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$. Still another way is: Every positive real number is less than some positive integer. If we regard this last statement as being a known property of real numbers, then the proof of Theorem 1 is quite easy. Suppose, indeed, that a satisfies the hypotheses of the theorem, but that $a > 0$. Then $\frac{1}{a}$ is a positive number, and there is a positive integer n such that $\frac{1}{a} < n$, by what we have just said. For this n, $1 < na$, contradicting the hypothesis $na \leq 1$. Therefore the assumption $a > 0$ is false. Since $a > 0$ or $a = 0$ by hypothesis, and the first is false, the second must be true.

We can now prove some rather obvious results which are usually assumed implicitly in customary treatments. They are, in fact, somewhat less obvious than some of the theorems that Euclid took the trouble to prove (e.g., the theorem that vertical angles are congruent). It is interesting to contemplate what the situation might have been if Euclid had decided that these were worthy of statement and proof. Perhaps schoolboys for centuries would have studied and proved:

Theorem 2. The area of a point is 0.

Proof. Let S be a unit square plus its interior. By Postulate 4, $A(S) = 1$. Let n be an arbitrary positive integer, and choose n points $P_1, P_2, \ldots, P_n$ in S. If T is the set $\{P_1, \ldots, P_n\}$, then by Postulate 2 (rather, by the generalization of Postulate 2 to n disjoint sets), we have $A(T) = A(P_1) + A(P_2) + \ldots + A(P_n)$. Now any two one-point sets are congruent, so by Postulate 3, $A(P_1) = A(P_2) = \ldots = A(P_n)$, and $A(T) = nA(P_1)$. Let R be all of S except for the points of T. Then R and T have no points in common and their union is S. By Postulate 2,

$$A(T) + A(R) = A(S).$$

By Postulate 1, $A(R) \geq 0$. Therefore

$$A(T) \leq A(S).$$

Substituting 1 for A(S) and $nA(P_1)$ for A(T), we get

$$nA(P_1) \leq 1.$$

In Theorem 1, we may take $a = A(P_1)$, since $A(P_1)$ is non-negative by Postulate 1. Therefore $a = 0$, that is, $A(P_1) = 0$. Since every point is congruent to P, A(P) = 0 for every point P, by Postulate 3.

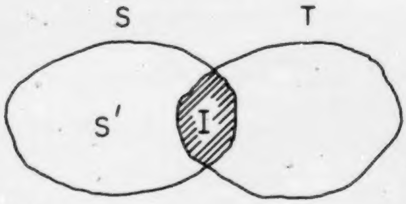
Observe that in the proof of Theorem 2, we proved and made use of a special case of

Theorem 3. If T is a subset of the bounded set S, then $A(T) \leq A(S)$.

The proof may be left to the reader.

Now we state a useful theorem which is similar to Postulate 2, but which has a weaker hypothesis.

Theorem 4. If S and T are bounded sets, V is the union of S and T, and I is the intersection of S and T, then $A(V) = A(S) + A(T) - A(I)$.



Proof: Let S' be the part of S not in T. Then the union of S' and I is S, and S' and I are disjoint. By Postulate 2,

$$A(S) = A(S') + A(I).$$

Also, the union of S' and T is V, and S' and T are disjoint. By Postulate 2,

$$A(V) = A(S') + A(T).$$

Therefore

$$A(V) = A(S) = A(I) + A(T)$$
$$= A(S) + A(T) - A(I) .$$

Theorem 5. If S and T are bounded sets and V is their union, then

$$A(V) \leq A(S) + A(T).$$

The proof follows from Theorem 4 on observing that $A(I) \geq 0$, by Postulate 1.

Theorem 6. If $S_1, S_2, \ldots, S_n$ are bounded sets and V is their union, then

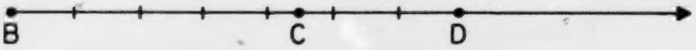
$$A(V) \leq A(S_1) + A(S_2) + \ldots + A(S_n).$$

The proof follows from Theorem 5 by induction.

Next, we prove another "obvious" theorem.

Theorem 7. The area of a segment is 0.

Proof. Let $\overline{BC}$ be a given segment, of length k. There is a natural number m such that $k \leq m$. On the ray $\overrightarrow{BC}$,



let D be the point such that $BD = m$. To prove that $A(\overline{BC}) = 0$ it is sufficient to show that $A(\overline{BD}) = 0$, by Postulate 1 and Theorem 3. Now $\overline{BD}$ is the union of m segments $S_1, \ldots, S_m$ of length 1. These segments are not disjoint, but we can still apply Theorem 6 to get

$$A(\overline{BD}) = A(S_1) + \ldots + A(S_m)$$
$$= m\, A(S_1),$$

since $S_1, \ldots, S_m$ are all congruent. Therefore it is sufficient to show that a segment of length 1 has area 0. The proof of this proceeds as in Theorem 2, by fitting an arbitrary number n of disjoint unit segments within a unit square. We omit the details.

We are now in a position to prove that the boundary of a polygonal region (defined in Chap. 11) has no influence on its area.

Theorem 8. Let R be a polygonal region and let R' be the same region with all or part of the boundary removed. Then $A(R') = A(R)$.

Proof: Let R_0 be the region R with all of the boundary removed. Then R_0 is contained in R' and R' is contained in R. Therefore

$$A(R_0) \leq A(R') \leq A(R),$$

by Theorem 3. It is sufficient to show that $A(R_0) = A(R)$. Let B be the boundary, consisting of a finite number of segments. By an application of Theorem 6, Theorem 7, and Postulate 1, we find that $A(B) = 0$. But R is the union of the disjoint sets R_0 and B, so

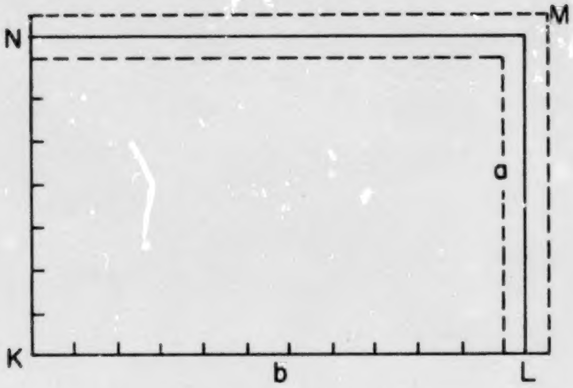
$$A(R) = A(R_0) + A(B) = A(R_0),$$

and the proof is complete.

Postulate 19 of the text now follows readily, since the overlap of the two regions R_1 and R_2 consists of a finite number of points and segments, and the area of the overlap is 0. We state Postulate 19 as a theorem, but omit the proof.

Theorem 9. Suppose that the polygonal region R is the union of two polygonal regions R_1 and R_2, which intersect only in edges and vertices. Then $A(R) = A(R_1) + A(R_2)$.

Now consider a rectangle R of base b and altitude a. We are aiming at a proof that $A(R) = ab$, this being Postulate 20 of the text.



Choose an arbitrary positive integer n, and determine p and q, also positive integers, by the conditions

$$\frac{p-1}{n} < b \leq \frac{p}{n},$$

$$\frac{q-1}{n} < a \leq \frac{q}{n}.$$

Starting at K, lay off p segments of length $\frac{1}{n}$ along ray $\overrightarrow{KL}$ and q segments of length $\frac{1}{n}$ along ray $\overrightarrow{KN}$. Then L is on the p-th segment on $\overrightarrow{KL}$ and N is on the q-th segment on $\overrightarrow{KN}$. The rectangular region R is now enclosed between two rectangular regions S and T, where S has dimensions $\frac{p-1}{n}$ and $\frac{q-1}{n}$, T has dimensions $\frac{p}{n}$ and $\frac{q}{n}$. Therefore

$$A(S) \leq A(R) \leq A(T).$$

Now S consists of $(p-1)(q-1)$ square regions of side $\frac{1}{n}$, and T consists of pq square regions of side $\frac{1}{n}$. If the area of one of these square regions is A_n, then

$$A(S) = (p-1)(q-1) A_n,$$

$$A(T) = pq A_n,$$

so

$$(p-1)(q-1) A_n \leq A(R) \leq pq A_n.$$

It remains to compute A_n and then $A(R)$. But a unit square, whose area is 1, can be split up into n^2 squares of side $\frac{1}{n}$, so

$$1 = n^2 A_n,$$

$$A_n = \frac{1}{n^2}$$

Therefore

$$(p-1)\,(q-1)\cdot \frac{1}{n^2} \leq A(R) \leq pq\cdot \frac{1}{n^2}\ .$$

Now, from the conditions determining p and q

$$\frac{p-1}{n}\cdot\frac{q-1}{n} \leq ab \leq \frac{p}{n}\cdot\frac{q}{n}\ .$$

The two fixed numbers $A(R)$ and ab both lie in the interval with endpoints $\frac{p-1}{n}\cdot\frac{q-1}{n}$, $\frac{p}{n}\cdot\frac{q}{n}$, so the absolute value of their difference is at most equal to the length of the interval:

$$|A(R) - ab| \leq \frac{p}{n}\cdot\frac{q}{n} - \frac{(p-1)}{n}\cdot\frac{(q-1)}{n}\ ,$$

$$|A(R) - ab| \leq \frac{p - q - 1}{n^2}\ .$$

Since $\frac{p}{n^2}$ is approximately $\frac{b}{n}$ and $\frac{q}{n^2}$ is approximately $\frac{a}{n}$, the right side is approximately $\frac{1}{n}(a+b)$, which is very small if n is large. An application of Theorem 1 to the fixed non-negative number $|A(R) - ab|/(a + b)$ would then yield that this number is 0. To make this argument precise, choose n so large that $\frac{1}{n} \leq a$ and $\frac{1}{n} \leq b$. Then $\frac{p-1}{n} < b$ implies that

$$\frac{p}{n} < b + \frac{1}{n} \leq 2b,$$

and $\frac{q-1}{n} < a$ implies that

$$\frac{q}{n} < a + \frac{1}{n} \leq 2a\ .$$

Therefore

$$\frac{p + q - 1}{n^2} < \frac{p + q}{n^2} = \frac{1}{n} \left(\frac{p}{n} - \frac{q}{n}\right) \leq \frac{2a + 2b}{n} .$$

Combining this with our previous inequality, we get

$$|A(R) - ab| \leq \frac{2a + 2b}{n} ,$$

or

$$n \cdot \frac{|A(R) - ab|}{2a + 2b} \leq 1,$$

for all sufficiently large positive integers n, and therefore for all n. By Theorem 1,

$$\frac{|A(R) - ab|}{2a + 2b}$$

is 0, so $A(R) = ab$. This completes the proof of

Theorem 10. The area of a rectangle is the product of its base and altitude.

We have now reached our goal of establishing Postulates 17-20 of the text from our system of Postulates 1-4. This may not seem like a great accomplishment if we are interested in polygonal regions only, but it permits the evaluation of areas of other regions without the necessity of making ad hoc extensions of the domain of the area function at a later stage. It provides us with an excellent example of the power of deductive reasoning. Finally, the transition from here to the integral calculus is a smooth and natural one. For example, the calculation of the area under the curve $y = x^n$, for all integers n (including $n = -1$) can be carried out on the basis of this development, without any reference to the differential calculus.

Miniature Geometries.

"Talks to the Teacher"

1. Preamble. In a given set of postulates for a special part of mathematics, it is hardly to be expected that the laws of classical logic, the rules of grammar and a definition of all the terms be included. We recognize their need but assume them whenever used. We also assume that the reader is familiar with the usual laws of arithmetic and algebra that may be used. Indeed there may be other needed logical assumptions that are overlooked so that the emphasis may be placed upon the particular topic under immediate discussion, and the postulates will be confined to those that have an immediate geometric use.

2. Characteristics of a Postulate System. What postulates should we make? There is no definite answer to this question. The answer depends upon the audience and upon the purpose and the preferences (or prejudices) of the individual. However, there are some desirable characteristics of the postulate system, which we proceed to discuss. We may not be able to attain all of them, and may have to make some compromises.

(1) Simplicity.* The postulates should be simple, that is, easily understood by the audience for which they are intended. But simple is a relative term, and depends upon the limited experience of the audience.

(2) Paucity. It may be desirable to have only a few undefined entities and relations and to make only a few assumptions about them. It may be necessary to sacrifice these characteristics to gain simplicity of understanding. Most texts on plane geometry for

*See Nelson Goodman, The Test of Simplicity, Science, Oct. 31, 1958, Vol. 128.

beginning students do sacrifice these characteristics, and some texts over-do it to avoid proving converses, especially if the method of proof by contradiction is needed. This puts a high premium on factual geometry as against logical geometry. It is not my purpose here to condemn or commend this point of view. It all depends upon the audience and the purpose of the text, but it may be very difficult to determine (except by the Rule of Authority) whether the system satisfies the next characteristic.

(3) Consistency. The postulate system should be consistent. It should not be self-contradictory. This part may be easy to determine. For example, we would not want to include two assumptions such as (A) Two lines in the same plane always have a point in common (Projective Geometry) and (B) There are lines in the same plane that have no point in common (Euclidean Geometry). But more is needed. The postulate system should never lead to a contradiction. This it may be difficult to determine or impossible to determine. We seldom know all the consequences of the postulate system, and in that case the proof of absolute consistency may not be possible within the system itself. (This is Gödel's great contribution to the subject of logic.) We content ourselves with relative consistency. If we can give at least one interpretation of the undefined terms based upon our experiences or experiments for which we grant all the assumptions are true, we are satisfied. We call such an interpretation a model. In the case of a simple system such as that for a miniature geometry, the construction of such models may be possible, and indeed in more than one way. In a complex postulate system, such as that needed for all of Euclidean

Geometry, logically developed, this may be extremely difficult. If we have more than one model for the same system so that we can find a correspondence connecting every entity and relation of one model with an entity and relation of the other model, that is, put the models into one-to-one correspondence, we say the models are isomorphic. We shall do this for some of our miniature geometries. But for more complex geometric systems, we may not have more than one model. The relative consistency of Euclidean Geometry is proved (but it is much too difficult for us to do it) by using arithmetic as a model, and showing it is possible to put Euclidean Geometry into one-to-one correspondence with arithmetic logically developed. Since we have never found a contradiction in arithmetic, we are content to say Euclidean geometry is as consistent as arithmetic. If we wish to prove a Non-Euclidean geometry is relatively consistent, we find a model (interpretation) within Euclidean geometry for it and after that is done (it is not an easy task and is beyond our intent), we know Non-Euclidean geometry is consistent if Euclidean geometry is. This is not the only way it can be done, for arithmetic (algebraic) methods are also available.

(4) Independence. It may be desirable to have all the postulates independent, especially if we are seeking models. By that we mean that the postulate system is such that no postulate can be derived from the others. The arguments present in (2) above are again applicable. In a given postulate system, it may be possible to prove that some of ghe assumptions could be derived from others, but it may be so difficult that it is a task to be avoided. However, it is not really difficult to prove: "Two distinct lines

cannot have more than one point in common" from the assumption: "There is one and only one line that contains two distinct points". The method of contradiction is used, and this points out the essential importance of this method of proof if we wish to make good use of our assumptions of logic. The independence of all the postulates of a system is most readily found in terms of models. If we can find a model that satisfies all but one of the postulates and denies that one, then that particular postulate is independent of the others. If we can do this for each postulate in turn, then the postulates form an independent system.

(5) Completeness. A postulate system for Euclidean geometry or any other special geometry we wish to discuss, should also be complete. That is, we must include enough postulates to prove all the theorems we wish to prove. This topic will not be discussed in detail here; it is enough to include a warning not to overlook tacit assumptions as Euclid* and his imitators did.

We illustrate various ideas mentioned above by confining our attention to incidence properties alone and make no attempt to discuss postulates of measure or separation, but do recognize that parallelism is essentially an incidence property. First we confine our attention to three types of miniature geometries which contain only a finite number of points and lines:

I. A three point - three line geometry; II. A four point - six line geometry; III. A seven point - seven line geometry.

After that we illustrate the incidence properties of Hyperbolic Geometry by considering two models in which the number of points on a line is infinite and where we change the parallel postulate from its usual Euclidean form.

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*See Felix Klein, Elementary Mathematics from an Advanced Standpoint; Meserve, The Foundations of Geometry, p.230-231; Wilder, Foundations of Mathematics, Chapt. I,2.

3. A Three Point Geometry.

Undefined: point, line on.

Concerning these undefined terms, we make the following four postulates:

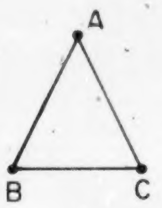
P1. There exist three and only three distinct points.

P2. On two distinct points there is (1) one and (2) only one line.

P3. Not all points are on the same line.

P4. On two distinct lines there is at least one point.

As far as consistency is concerned, there does not seem to be any direct contradiction. The relative consistency of the system is accepted on the basis of any one of the following three isomorphic models.



(a) The usual model of a triangle, consisting of three non-collinear points, but here a line contains only two points. The line segments of a more complete geometry are merely drawn to point out the three pairs of points. A line is merely a set of two points. It is easy to observe that Postulates P1 -- P4 are all satisfied.

(b) A group of three boys forming committees of two in all possible ways. If the boys are called A, B, C, the committees are the three pairs (A,B), (B,C), (C,A). If the postulates are read with 'boy' replacing 'point', 'committee' replacing 'line' and 'member of' replacing 'on', with possible changes in language to preserve the meaning, it is easy to see P1, P2, P3 are obviously satisfied by the way the committees were formed. A simple observation of the three committees checks P4.

(c) Points are interpreted as the special ordered number triples (x,y,z): A(1,0,0), B(0,1,0), C(0,0,1). Lines are interpreted as the special equations $x = 0$, $y = 0$, $z = 0$. A 'point' is 'on' a 'line' if its coordinates satisfy the equation of the line.

P1 follows from our choice of coordinates.

P2 must be verified: A(1,0,0) and B(0,1,0) are both on $z = 0$ but not both are on $x = 0$ or $y = 0$. A similar verification is needed for the other pairs of points.

P3: The point A(1,0,0) does not satisfy the equation of the line BC:$x = 0$.

P4: There are three distinct pairs of lines (i) $x = 0$, $y = 0$; (ii) $y = 0$, $z = 0$; (iii) $z = 0$, $x = 0$. It is easy to verify that C(0,0,1), A(1,0,0), B(0,1,0) lie on the pairs (i), (ii), (iii) respectively.

We prove three theorems directly from the postulates without a model. For heuristic purposes any one of the models could be used.

Theorem 1. On two distinct lines there is not more than one point.

Proof. If two lines had two distinct points in common, then Postulate P2 would be contradicted. Hence Theorem 1 is true.

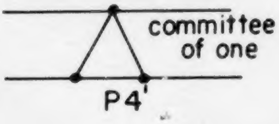
Theorem 2. There exist three and only three lines.

Proof. Since there are three and only three points (P1), there are only three pairs of points: (A,B), (B,C), (C,A). Each such pair determines one and only one line (P2). These lines are all distinct (P3). Hence there are three and only three lines.

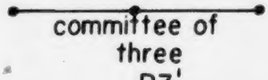
Theorem 3. Not all lines are on the same point.

Proof. There are three and only three lines (A,B), (B,C), (C,A), (Theorem 2). The first and third are on the point A, but this point is not on the line (B,C) because of P3. A similar argument concerning the points B and C completes the proof.

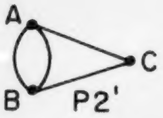
Of course all three of these theorems could have been verified in any model. That is, we could have taken them as postulates too, but then the system would not have been an independent one. To demonstrate the independence of the original system P1 - P4 we use geometric models but either of the other models could be used equally as well. We use the notation P4' to indicate that P4 is denied but P1, P2, P3 are satisfied. Similar meanings are given to P3', P2', and P1'. The model P4' is constructed by adding a fourth line (denying Theorem 2) in such a way that there are two lines which have no point in common. This denies P4, but the other postulates are satisfied. In the model P3', all three points are on the same line and the other postulates may be verified. In the model P2', there are two lines which contain both A and B. In terms of the committee interpretation you may think of A and B both being on two distinct committees, say the Finance Committee and the Custodian Committee. The model for P1' is not shown here. It must contain more than three points. The smallest such model which will



P4'



P3'



P2'

also satisfy the other axioms is the model for a seven point geometry to be discussed in Section 5. After that model is presented the proof of the independence of the system P1 - P4 will be complete.

4. *A Four Point Geometry*. Again *point*, *line*, and *on* are undefined. To distinguish the postulates from those just used we use the letter Q.

Q1. *There exist four and only four distinct points*.

Q2. *On two distinct points there is one and only one line*. (P2)

Q3. *Every line contains two and only two points*.

Theorem 1. *There exist six and only six lines*.

Proof. The number of pairs of points is the number of combinations of four things taken two at a time, ${}_4C_2 = \frac{4\cdot 3}{2} = 6$ (Q1) and this is the number of lines (Q2). These lines are all distinct, (Q3). Hence the theorem is proved.

If we call the points O, A, B, C, the lines are represented by the point pairs (O,A); (O,B); (O,C); (A,B); (A,C); (B,C).

Definition. *Two lines are parallel if they have no point in common*.

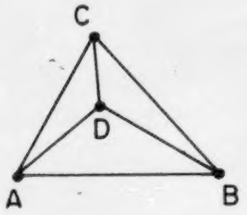
Note that the word *parallel* is used in a very special sense. No concept of a plane has yet been introduced.

Theorem 2. *Through a given point not on a given line there is one and only one line parallel to the given line*.

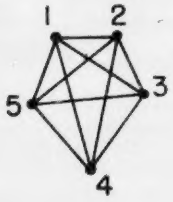
Proof. A given point, say A, lies on three and only three lines and these lines are distinct (Q1, Q2, Q3). If we pick one of these lines, say AO, neither of the remaining points, B and C, can lie on it (Q3), and hence the two lines have no point in common or are *parallel* by definition.

Several models of this geometry are available. The two-member committee model is quite apparent. Each member is on three committees but there is always a unique second committee that can meet while this member is engaged in committee business.

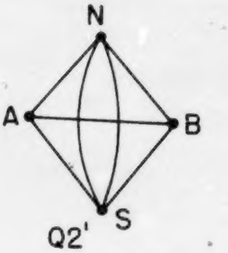
In order to present geometric models, we imagine the model to be embedded in ordinary Euclidean geometry and then abstract from the diagram those features that are wanted. One such model is that of a complete quadrangle (a term borrowed from projective geometry) which consists of four points, no three collinear, and the six lines which they determine by pairs. Of course you must recognize that our line is only a point-pair. It is easy to verify that Postulates Q1, Q2, Q3 are all satisfied. Models Q1', Q2', Q3', needed to prove the postulate are independent, are more or less self-explanatory.



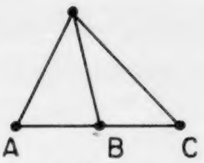
Q1, Q2, Q3



Q1'



Q2'



Q3'

If the model Q2' bothers you, think of it in terms of a diagram drawn on a sphere with N and S being the poles, or if you know something of chemical bonds, think of it in terms of a double bond between N and S, and all the rest as single bonds.

The figure for Q1, Q2, Q3 could be imagined in ordinary 3-space thus forming a tetrahedron. Indeed we could then add additional postulates.

Undefined: plane.

Q4. On three points there is one and only one plane.

If we think entirely in terms of plane geometry each of the models already drawn also satisfy Q4.

Q5. Every plane contains three and only three points.

None of the models of plane geometry satisfy this axiom, which, however, is satisfied by the tetrahedron model. That is, the tetrahedron model satisfies all five postulates Q1 - Q5. It is possible to present models in 3-space to prove the independence of these five postulates but this will not be done here, but the reader is urged to try his hand at it.

The committee interpretation of this enlarged system takes into account three-member committees as well as two-member committees. Our tetrahedron model is for a four point - six line - four plane geometry.

Let us return to the system Q1, Q2, Q3 and its two geometric interpretations and discuss algebraic systems isomorphic to them. For the complete quadrangle model, we consider points as the special ordered number triples (x, y, z): A(1,0,0); B(0,1,0); C(0,0,1); O(1,1,1). As lines we take the six equations $x = 0$, $y = 0$, $z = 0$, $x = y$, $y = z$, $z = x$. We say a point is on a line if its coordinates satisfy the equation of the line.

Q1 is satisfied by the way coordinates were introduced. It is now possible to verify Q2 and Q3. There are six pairs of points and it is possible to show that any pair lies on one and only one line and this line contains neither of the other points. For example B(0,1,0) and C(0,0,1) satisfy the equation $x = 0$, but neither A(1,0,0) nor O(1,1,1) do; B(0,1,0) and O(1,1,1) satisfy the equation $x = z$, but neither of the points $A(1,0,0)$ or $C(0,0,1)$ do. Similarly, for the four other pairs.

For the tetrahedron model, we consider points as the special ordered number triples (x, y, z): $A(1,0,0)$; $B(0,1,0)$; $C(0,0,1)$ and $O(0,0,0)$. (Note the difference between the two models.) As the lines we consider the six pairs of equations which can be formed from the four equations $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$. (These are the equations of the four planes.) Q1 is satisfied by the way coordinates were introduced. It is now possible to verify Q2 and Q3. For example $B(0,1,0)$ and $C(0,0,1)$ satisfy the two equations $x = 0$, $x + y + z = 1$, but both do not lie on either $y = 0$ or $z = 0$. A similar analysis can be given for every other pair of points. In this algebraic model, postulates Q4 and Q5 also may be verified.

5. A Seven Point Geometry. As mentioned earlier this geometry is one that denies the existence of only three points but satisfies P2, P3, P4 of the three-point geometry. We repeat these postulates for convenience of reference. The essential distinction between this geometry and those already discussed is that every line contains three and only three points. It is necessary to include a postulate which guarantees there is at least one line.

Undefined: point, line, on

P2. On two distinct points there is one and only one line.

P3. Not all points are on the same line.

P4. On two distinct lines there is at least one point.

P5. There exists at least one line.

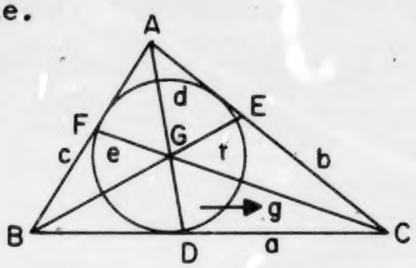
P6. Every line is on at least three points.

P7. No line is on more than three points.

Of course P6 and P7 could be put together to say that every line is on three and only three points.

We construct a special model for this postulate system by selecting seven distinct points, which we call A, B, C, D, E, F, G. We define seven and only seven lines, each being a set of three points, by means of the following table.

A	B	C	D	E	F	G
B	C	E	A	G	D	F
F	D	A	G	B	E	C
c	a	b	d	e	g	f



It is not our purpose to discuss the many theorems that can be proved from this postulate system, but to point out several interpretations of it. It may bother you a bit to call (D,E,F) a line, but it is a line by definition just as much as the triple (A,B,F) is a line. Of course this geometry is not like the Euclidean geometry of your experience -- it is a finite projective geometry where we have considered only incidence properties. However, its interpretation as a group of seven persons and seven committees of three and only three members is also available. Since we set up the model by definition (committee aspect) and then drew a diagram to correspond, we must verify all the postulates P2 to P7. This may be long in detail but it is not difficult. There are 21 pairs of points $({}_7C_2 = \frac{7\cdot 6}{2})$ and 21 pairs of lines, but an examination of the table shows that each row contains each letter once and only once, and each letter is in three and only three columns, and this will simplify the details. It is merely time consuming to verify all the postulates; these postulates are all satisfied in the geometric model. To verify P4, for example, from the table, it is necessary to consider 21 pairs of lines, and indeed it is easy to verify not only that each pair has a point in common (there are no pairs of parallel lines) but only one point in common.

The results can be tabulated as follows

c	a	b	d	e	g	f
b	e	f	a	g	c	d
d	c	a	g	b	f	e
A	B	C	D	E	F	G

Not only may we verify P2 - P7 in this way, but also the dual of each of these statements. The dual is obtained by interchanging the words point and line wherever they appear. For example the dual statement to P6 and P7 combined would read:

D6,7. Every point is on three and only three lines. This is easily verified from the defining table.

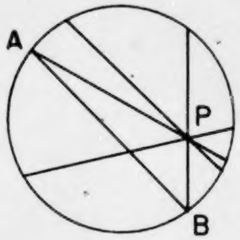
The algebraic isomorphism for this geometry consists of the following assignments of coordinates to points and equations to lines:

$A(1,0,0)$; $B(0,1,0)$; $C(0,0,1)$; $D(0,1,1)$; $E(1,0,1)$; $F(1,1,0)$; $G(1,1,1)$
a: $x = 0$; b: $y = 0$; c: $z = 0$; d: $y = z$; e: $x = z$; f: $x = y$;
g: $x + y + z = 2$.

All the postulates could be verified purely algebraically. For example, $D(0,1,1)$ and $E(1,0,1)$ both lie on the line $x + y + z = 2$, but not both are on any other line. The line $x = y$ contains the three points $C(0,0,1)$, $G(1,1,1)$, $F(1,1,0)$ but no other point. That is enough to give the general idea.

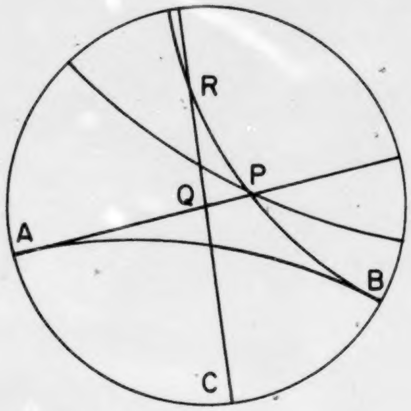
6. Models for a Hyperbolic Geometry. In order to discuss such a model, it will be embedded in a Euclidean plane. Hence we assume that the postulates of Euclidean geometry as stated in the S.M.S.G. text have been made and Euclidean geometry has been developed. We will use the terms point, line, plane, and circle as developed in such a treatment. The corresponding words placed in quotes will stand for entities in a new geometry, and will be defined by means of Euclidean terms. In this way we will obtain models to illustrate some of the incidence properties of Hyperbolic geometry.

The first model is often called a projective model, but the explanation of the term is beyond our present means. Consider a (large) circle. We define a "point" of our new geometry to be a point in the interior of the circle; a "line" is a chord of this circle without its end-points; the "plane" is the interior of the circle. It is easy to observe that two "lines" may or may not intersect. If two chords of the circle intersect on the circle, we say that the corresponding "lines" are "parallel". Note that there is a definite distinction between two "lines" being "parallel" and two "lines" not intersecting. It is also easy to observe that through a given "point" P, there are two "lines", $\overline{PA}$ and $\overline{PB}$, which are "parallel" to the "line" $\overline{AB}$, and that there are an infinite number of "lines" through P that do not intersect the "line" $\overline{AB}$.



In the above model length and angular measure are distorted, and a study of projective geometry is needed to discuss the model.

There is a model, called Poincaire's Universe, where length is distorted but angular measure is not (but no proof is intended). To understand this model some knowledge of orthogonal circles in Euclidean Geometry is required, and the corresponding theorems are not usually presented in an introductory course in plane geometry. We state the necessary definitions and theorems (without proof).



Two circles are orthogonal if their angle of intersection is a right angle. By the angle of intersection of two circles we mean the angle between the tangent lines drawn at a common point.

Through two points there is one and only one circle (or line) orthogonal to a given circle.

In the Poincaire model, a "point" is again a point inside a given circle C, and the "plane" is the set of all points in the interior of the circle. A "line" is either a diameter of the circle C, without its end-points, or that part of a circle orthogonal to the circle C which lies inside C. We note, therefore, that through two "points" there is one and only one "line". Two "lines" are said to be "parallel" if their corresponding diameters or circles intersect on C. It is again easy to observe that through a given "point" P, there are two "lines" $\overset{\frown}{PA}$ and $\overset{\frown}{PB}$, which are "parallel" to the "line" $\overset{\frown}{AB}$, and that there are an infinite number of "lines" through P that do not intersect the "line" $\overset{\frown}{AB}$. One more idea may be observed in this diagram (based on the assumption that angular measure is not distorted).

"The sum of the measures of the "angles" of a "triangle" such as Δ AOB or Δ PQB is less than 180."

A more detailed study of the geometry of the circle in the Euclidean plane, including a study of the concept of cross-ratio is needed to carry the discussion further. Some further results and suggestions or indications of ideas that might be investigated can be found in Eves and Newson, Introduction to Foundations and Fundamental Concepts of Mathematics.

Introduction to Non-Euclidean Geometry

About one hundred and fifty years ago, a revolution in mathematical thought began with the discovery of a geometrical theory which differed from the classical theory of space formulated by Euclid about 300 B.C. Euclid's geometry text, the Elements, was the finest example of deductive thinking the human race had known, and had been so considered for two thousand years. It was believed to be a perfectly accurate description of physical space, and at the same time, the only way in which the human mind could conceive space. It is no small wonder then that the development of theories of non-Euclidean geometry had an impact on mathematical thought comparable to that of Darwin in biology, Copernicus in astronomy or Einstein in physics.

How did this revolutionary change come about? Strangely enough it may be considered to have had its origin in Euclid's text. Although he lists his postulates at the beginning, he refrains from employing one of them until he can go no farther without it. This is the famous fifth postulate which we may state in equivalend form as

Euclid's Parallel Postulate. If point P is not on line L , there exists only one line through P which is parallel to L .
It seems probable that Euclid deferred the introduction of the fifth postulate because he considered it more complex and harder to grasp than his other postulates.

The consequences of introducing Euclid's parallel postulate are almost phenomenal. Using it we get in sequence:

1. The alternate interior angle theorem for parallel lines;
2. The sum of the measures of the angles of a triangle is 180;
3. Parallel lines are everywhere equidistant;
4. The existence of rectangles of preassigned dimensions.

As remote but recognizable consequences of Euclid's parallel postulate, we have:

5. The familiar theory of area in terms of square units which in effect reduces any plane figure to an equivalent rectangle;

6. The familiar theory of similarity;

7. The Pythagorean Theorem, the proof of which depends upon the theory of similar triangles.

It is hard to see how any of these important results could be proved without recourse to Euclid's parallel postulate or an equivalent assumption.

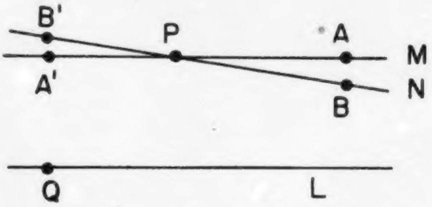
There is no explicit evidence that Euclid considered the fifth postulate an improper assumption in his basis for geometry. But generations of mathematicians for over 2000 years were dissatisfied with it, and worked hard and long in attempts to deduce it as a theorem from the other seemingly simpler postulates. Right up to the beginning of the 19th century able mathematicians convinced themselves that they had settled the problem only to have flaws discovered in their work. Sometimes they employed the principle of the indirect method and developed elaborate and subtle arguments to prove that the denial of Euclid's parallel postulate would force one into a contradiction. None of these arguments stood up under analysis. Finally early in the 19th century, J. Bolyai (1802-1860) a Hungarian army officer, and N. I. Lobachevsky (1793-1856) a Russian professor of mathematics at the University of Kazan, independently introduced theories of geometry based on a contradiction of Euclid's parallel postulate.

The purpose of this talk is to give an elementary introduction to the non-Euclidean theory of geometry which Bolyai and Lobachevsky created.

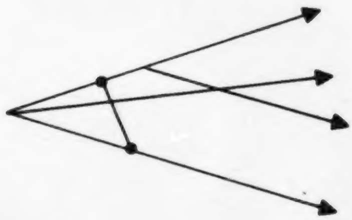
I. Two Non-Euclidean Theorems.

In this part we try to give you - without a long preliminary discussion - the flavor of non-Euclidean geometry. Our viewpoint is this: Suppose we consider the hypothesis that there are two lines parallel to a particular line through a particular point. What will follow? As a basis for our deductions we assume the postulates of Euclidean geometry except the parallel postulate, specifically Postulates 1,...,15 of the text.

Theorem 1. Let P be a point and L a line such that there are two lines through P each of which is parallel to L. Then L is wholly contained in the interior of some angle.



Proof. Let lines M and N contain P and be parallel to L . Then M and N separate the plane into four "parts" each of which is the interior of an angle. Specifically these parts or regions may be labelled as the interiors of the angles $\angle APB$, $\angle A'PB'$, $\angle A'PB$, $\angle APB'$ where P is between A and A' on M and P is between B and B' on N. Let Q be any point of L . Since L does not meet M or N , Q is not on M or N . So Q is one of the four angle interiors say the interior of $\angle A'PB$. Now where can L lie? Note that one of its points Q is in the interior of $\angle A'PB$ and that L does not meet the sides of the angle $\angle A'PB$. Clearly L is trapped inside $\angle A'PB$ - that is L is wholly contained in the interior of $\angle A'PB$ and the theorem is proved.



Observe how strange this is when compared with the Euclidean situation where only a part of a line can be contained in the interior of an angle, as indicated in the figure. But note - as always in mathematics - the inevitability of the result once the hypothesis is granted. You may say the argument is valid abstractly - but it doesn't correspond to physical reality.

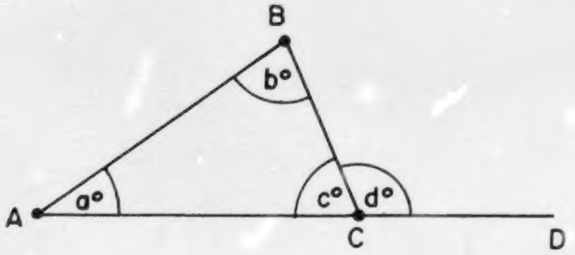
As you make a statement like this you begin to tread the path of the non-Euclidean geometers. All that one needs to think mathematically is a set of precisely stated assumptions (postulates) from which conclusions (theorems) can be derived by logical reasoning. Are these assumptions absolutely true when applied to the physical world? We don't really know. It is not our professional concern as mathematicians to answer the question. It lies in the domain of physicists, astronomers and surveyors. As human beings who work in mathematics we may like to feel that our theories are applicable to physical reality. But this doesn't require the absolute truth of our postulates or our theorems. When Euclidean geometry for example is applied by an architect or engineer or surveyor he doesn't require results which are absolutely correct - he might consider this a mirage. Rather he demands results correct to the degree of precision required by his problem - accuracy of one part in a hundred might be excellent in a pocket magnifying glass but one part in a million might be too rough for a far-ranging astronomical telescope.

Our first theorem indicated how positional or non-metrical properties in a non-Euclidean geometry might differ from our Euclidean expectations. Now we show how metrical properties - specifically the angle sum of a triangle - are altered when we change the parallel postulate.

Theorem 2. Let P be a point and L a line such that there are two lines through P each of which is parallel to L. Then there exists at least one triangle the sum of whose angle measures is less than 180.

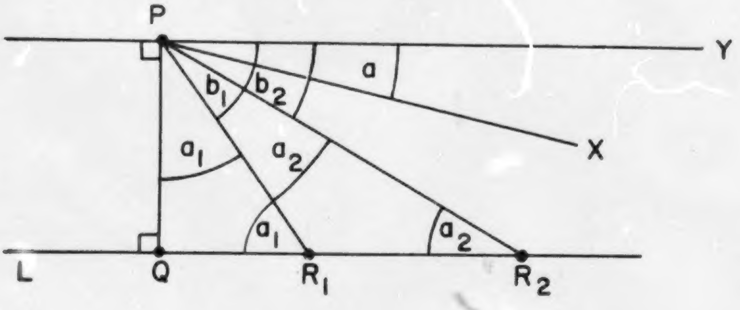
Proof. Suppose the theorem false. Then the sum of the angle measures of every triangle is greater than or equal to 180. From this we derive the

Lemma. The sum of the measures of two interior angles of a triangle is greater than or equal to the measure of the remote exterior angle.



Proof of Lemma. We have $a + b + c = 180$. Hence

$$a + b = 180 - c = d \ .$$



Continuation of Proof of Theorem. Let L be a line and P a point such that there are two lines through P parallel to L. Let line $\overleftrightarrow{PQ}$ be perpendicular to L at Q. Since there are two lines through P parallel to L one of these must make an acute angle with line $\overleftrightarrow{PQ}$. Suppose then line $\overleftrightarrow{PX}$ is parallel to L and makes an acute angle, $\angle QPX$, with line $\overleftrightarrow{PQ}$. Let line $\overleftrightarrow{PY}$ be perpendicular to line $\overleftrightarrow{PQ}$ with Y on the same side of line $\overleftrightarrow{PQ}$ as X. Let $m\angle YPX = a$; then $a \angle 90$. (Think of a as a small positive number say .1.) Now locate R_1 on L so that $QR_1 = PQ$ and R_1 is on the same side of $\overleftrightarrow{PQ}$ as X and Y. Draw segment $\overline{PR_1}$. Then ΔPQR_1 is isosceles so that $m\angle QPR_1 = m\angle QR_1P = a_1$. Since the exterior angle of ΔPQR_1 at Q is a right angle, the Lemma implies

$$a_1 + a_1 = 2a_1 \geq 90$$

and

$$a_1 \geq 45 .$$

Let $m\angle YPR_1 = b_1$. Then

$$b_1 + a_1 = 90$$

so that $$b_1 = 90 - a_1$$

and $$b_1 \leqq 45$$

Moreover $$b_1 > a \, .$$

Now we repeat the argument by constructing a new triangle. Extend segment $\overline{QR_1}$ to R_2 making $R_1R_2 = PR_1$. Draw $\overline{PR_2}$. Then ΔPR_1R_2 is isosceles, so that $m\angle R_1PR_2 = m\angle R_1R_2P = a_2$. By the Lemma

$$a_2 + a_2 = 2a_2 \geqq a_1 \quad .$$

So that $$2a_2 \geqq a_1 \geqq 45$$

and $$a_2 \geqq 45/2$$

Let $m\angle YPR_2 = b_2$ Then

$$b_2 + a_2 = b_1$$

$$b_2 = b_1 - a_2 \, .$$

Since $b_1 \leqq 45$ and $a_2 \geqq 45/2$ we have

$$b_2 \leqq 45/2$$

Moreover $$b_2 > a \, .$$

Continuing in this way we obtain a sequence of real numbers

$$b_1, b_2, b_3, \ldots$$

which are less than or equal to respectively

$$45, 45/2, 45/4, \ldots$$

but all of which are greater than the fixed positive number a This is impossible since repeated halving of 45 must eventually produce a number less than a. So our supposition is false and the theorem holds.

A proof of this type, though not difficult, may be unfamiliar and you may have to mull it over a bit to appreciate it better. In intuitive terms it is not very hard. There are two main points. First, the ray $\overrightarrow{PX}$ which doesn't meet L acts as a sort of boundary for the rays $\overrightarrow{PR_1}$, $\overrightarrow{PR_2}$,... which do meet L . Thus the angles $\angle YPR_1$, $\angle YPR_2$,.. have measures b_1, b_2,... which are greater than a. On the other hand (if the sum of the angle measures of every triangle is at least 180) we can pile up successive angles $\angle QPR_1$, $\angle R_1PR_2$,..., starting at ray $\overrightarrow{PQ}$, of measures at *least* 45, 45/2, 45/4,.. so that the angles $\angle YPR_1$,$\angle YPR_2$,... have measures at *most* 45, 45/2, 45/4,.... So we have a contradiction in that the angles $\angle YPR_1$, $\angle YPR_2$,. have measures which approach zero but are all greater than a fixed positive number a .

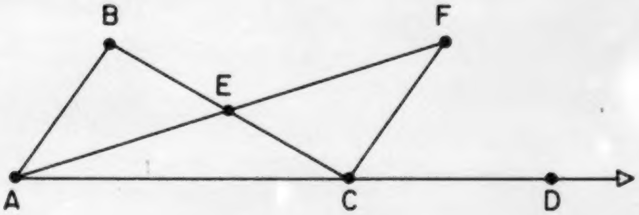
A final remark. You may object that we have not really justified that $\overrightarrow{PX}$ is a "boundary" for $\overrightarrow{PR_1}$, $\overrightarrow{PR_2}$,.... To take care of this observe that $\overrightarrow{PR_1}$ and $\overrightarrow{PX}$ are on the same side of line $\overleftrightarrow{PQ}$ Consequently one of them must fall inside the angle formed by $\overrightarrow{PQ}$ and the other Suppose $\overrightarrow{PX}$ fell inside $\angle QPR_2$. Then $\overrightarrow{PX}$ would meet line $\overleftrightarrow{QR_1}$ or L . Since this is impossible, $\overrightarrow{PR_1}$ must line inside $\angle QPX$. Similarly for $\overrightarrow{PR_2}$,....

II. Neutral Geometry.

We are using the term "neutral geometry" in this part to indicate that we are assuming neither Euclid's parallel postulate nor its contradictory. We shall merely deduce consequences of Euclid's postulates other than the parallel postulate, (specifically our discussions are based on Postulates 1, . . ,15 of the text). Our results then will hold in Euclidean geometry and in the non-Euclidean geometry of Bolyai and Lobachevsky since they are deducible from postualtes which are common to both theories. Our study is neutral also in the sense of avoiding controversy over the parallel postulate. Actually its study helps us to accept the idea of non-Euclidean geometry since it points up the fact that mathematically we have a more basic geometrical theory which can be definitized in either of two ways.

We proceed to derive some results in neutral geometry. Since you are familiar with so many striking and important theorems which do depend on Euclid's parallel postulate you might think that there are no interesting theorems in neutral geometry. However, this is not so. First we sketch the proof of a familiar and important theorem of Euclidean geometry whose proof does not depend on a parallel postualte (sec text p. 200 Theorem 7-1).

Theorem 1. An exterior angle of a triangle is larger than either remote interior angle.



Proof. Given ΔABC with exterior angle $\angle BCD$. We show $m\angle BCD$ is greater than $m\angle B$ and $m\angle A$. Let E be the midpoint of segment $\overline{BC}$ and let F be the point such that $AE = EF$ and E is between A and F. It follows that $\Delta BEA \cong \Delta CEF$ so that

$$m\angle B = m\angle ECF .$$

But

$$m\angle BCD = m\angle ECF + m\angle FCD .$$

Substituting $m\angle B$ for $m\angle ECF$ we have

$$m\angle BCD = m\angle B + m\angle FCD ,$$

so that

$$m\angle BCD > m\angle B .$$

The proof is completed as usual by applying the above argument to show that the vertical angle of $\angle BCD$ is larger than $\angle A$.

Corollary 1. The sum of the measures of two angles of a triangle is less than 180.

Proof. Given ΔABC we show $m\angle A + m\angle B < 180$. By the theorem $m\angle A$ is less than the measure of an exterior angle at B . Thus

$$m\angle A < 180 - m\angle B$$

so that

$$m\angle A + m\angle B < 180 .$$

This corollary is important since, without assuming a parallel postulate, it gives us information about the angles of a triangle. It tells us for example that a triangle can have at most one obtuse angle or at most one right angle.

Corollary 2. In a plane two lines are parallel if they are both perpendicular to the same line (compare text, Theorem 9-1).

Proof. The basic properties of perpendicular lines in Euclidean geometry are studies prior to the introduction of the parallel postulate, and so are part of (or are valid in) neutral geometry. Thus the familiar proof of the corollary is applicable: If the two lines met we would have, in a plane, two lines perpendicular to the same line at the same point. This is impossible and the lines can't meet

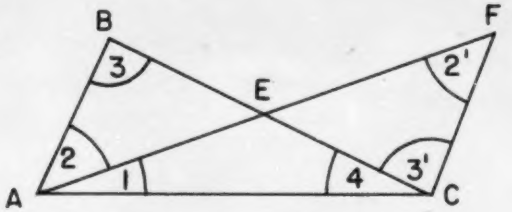
Corollary 3. Let L be a line, and let P be a point not on L. Then there is at least one line through P , parallel to L.

Proof. This follows from Corollary 2 by the familiar theorem on the existence of perpendiculars: Let $L_1 \perp L$ through P , and $L_2 \perp L_1$ through P. Then $L_2 // L$.

Observe that this familiar - almost hackneyed - discussion has yielded a very important principle: That parallel lines exist. More precisely, there exists at least one line parallel to a given line through an external point. And we got this result without assuming any parallel postulate! So the crucial point in our study of the theory of parallelism will be whether there is one, or more than one, line parallel to a given line through an external point.

To prove an important, and not sufficiently well known, theorem of Legendre (1752-1833) we introduce the following:

Lemma. Given ΔABC and $\angle A$. Then there exists a triangle $\Delta A_1B_1C_1$ such that: (a) it has the same angle measure as ΔABC; (b) $m\angle A_1 \leqq \frac{1}{2} m\angle A$.



Proof. We use the same construction as in Theorem 1 above. Let E be the midpoint of $\overline{BC}$ and let F satisfy $AE = EF$ and E is between A and F. Then $\Delta BEA \cong \Delta CEF$ and corresponding angles have equal measures. ΔAFC is the $\Delta A_1B_1C_1$ we are seeking. We have

$$\begin{aligned} m\angle A + m\angle B + m\angle C &= m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 \\ &= m\angle 1 = m\angle 2' + m\angle 3' + m\angle 4 \\ &= m\angle CAF + m\angle AFC + m\angle FCA. \end{aligned}$$

To complete the proof note

$$m\angle A = m\angle 1 + m\angle 2 = m\angle 1 + m\angle 2'$$

so that

$$m\angle A = m\angle CAF + m\angle AFC .$$

Hence one of the terms on the right is less than or equal to 1/2 the term on the left, that is $\frac{1}{2} m\angle A$. Consequently ΔAFC can be relabeled $\Delta A_1B_1C_1$ so as to make the theorem valid.

Note that since we have not assumed Euclid's parallel postulate we don't know that the angle measure sum is constant for all triangles. So the lemma is a significant result in that we can construct from a given triangle a new one with the same angle measure sum. In intuitive terms we can replace a triangle by a "slenderer" one without altering its angle measure sum. In effect the proof shows this by cutting off ΔABE from ΔABC and pasting it back on as ΔFCE.

Now we can prove the following remarkable theorem.

Theorem 2 (Legendre). The angle measure sum of any triangle is less than or equal to 180.

Proof. Suppose the contrary. Then there must exist a triangle, ΔABC, whose angle measure sum is $180 + p$, where p is a positive number. Now we apply the lemma. It tells us that there exists a slenderer triangle, $\Delta A_1B_1C_1$, whose angle measure sum also is $180 + p$ such that $m\angle A_1 \leqq \frac{1}{2} m\angle A$.

To fix our ideas let us say $p = 1$ and $m\angle A = 25$. Then

$$m\angle A_1 + m\angle B_1 + m\angle C_1 = 181 \quad \text{and} \quad m\angle A_1 \leqq 25/2 .$$

Pressing our advantage we reapply the lemma. So there is a still slenderer triangle, let us call it $\Delta A_2B_2C_2$, whose angle measure is $180 + p$ and $m\angle A_2 \leqq \frac{1}{2} m\angle A_1$. That is

$$m\angle A_2 + m\angle B_2 + m\angle C_2 = 181 \quad \text{and} \quad m\angle A_2 \leqq 25/4$$

Continuing in this way, we get a sequence of triangles each with angle measure sum 181 and with successive angles of measures no greater than

$$25,\ 25/2,\ 25/4,\ 25/8, \ldots .$$

To see this is impossible, consider $\Delta A_5B_5C_5$ for which $m\angle A_5 < p$.

We have

$$m\angle A_5 + m\angle B_5 + m\angle C_5 = 181 \quad \text{and} \quad m\angle A_5 \leqq 25/32 .$$

Certainly

$$m\angle A_5 < 1$$

but

$$m\angle B_5 + m\angle C_5 < 180$$

by Corollary 1 to Theorem 1.

Adding the inequalities

$$m\angle A_5 + m\angle B_5 + m\angle C_5 < 181 .$$

This contradiction implies our supposition false, and the theorem is established.

Note the point of the proof is to get a triangle so "slender", that is with one angle so small, that the triangle can't exist by Corollary 1 above It may now be instructive to write out the proof in general terms without assigning specific values to p and m$\angle$A .

Corollary. The angle measure sum of any quadrilateral is less than or equal to 360.

III. Do Rectangles Exist?

We continue to study neutral geometry, and are interested in whether a rectangle can exist in such a geometry, and what happens if it does. So most of our theorem will have the hypothesis that a rectangle exists We use freely the results of the last part on neutral geometry.

The existence of a rectangle in a geometry is not a trivial thing - imagine what Euclidean geometry would be like if you didn't have or couldn't use rectangles If you try to construct a rectangle you will find you are assuming Euclid's parallel postulate or one of its consequences, such as, the angle measure sum of a triangle is 180.

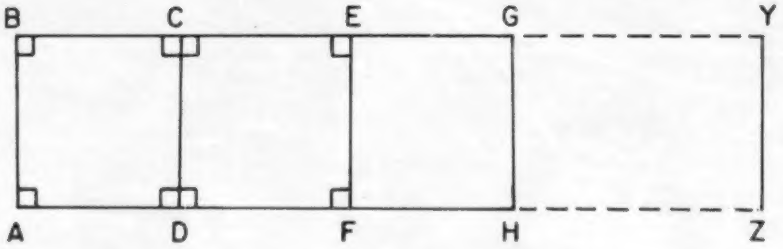
First to avoid ambiguity we formally define rectangle as we shall use the term:

Definition. A (plane) quadrilateral is called a rectangle if each of its angles is a right angle.

Notice that since we are operating in neutral geometry and have not assumed Euclid's parallel postulate, we can't automatically apply familiar Euclidean propositions, such as (1) the opposite sides of a rectangle are parallel, or (2) that they are equal in length, or (3) that a diagonal divides a rectangle into two congruent triangles. If we want to assert any of these results we will have to prove them from our definition without assuming a parallel postulate. For example (1) is immediate by Theorem 1 Corollary 2 of the preceding part.

Theorem 1. If one particular rectangle exists then a rectangle exists with an arbitrarily large side.

Restatement. Suppose a rectangle ABCD exists and x is a given positive real number. Then there exists a rectangle with one side of length greater than x



Proof. We use ABCD as a "building block" to construct the desired rectangle. Construct a quadrilateral DCEF congruent to ABCD , so that $\overline{EF}$ and $\overline{AB}$ are on opposite sides of line $\overleftrightarrow{CD}$. Then DCER is a rectangle. Moreover B, C, E lie on a line by a familiar perpendicularity property Similarly A, D, F are collinear. So ABCEFD is a quadrilateral ABEF and consequently a rectangle. Note ABEF has the property that

$$AF = 2\ AD .$$

Similarly we construct FEGH a congruent replica of ABCD so that $\overline{GH}$ and $\overline{AB}$ are on opposite sides of line $\overleftrightarrow{EF}$. And we see that ABGH is a rectangle such that

$$AH = 3AD .$$

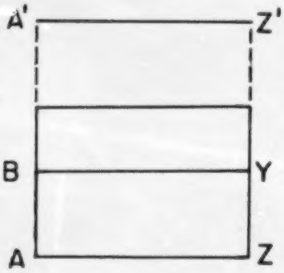
Continuing in this way we can construct a rectangle ABYZ such that

$$AZ = nAD$$

for each positive integer n . Now choose n so big that $nAD > x$. Then ABYZ satisfies the conditions of our theorem.

Corollary. If one particular rectangle exists, then a rectangle exists with two arbitrarily large adjacent sides.

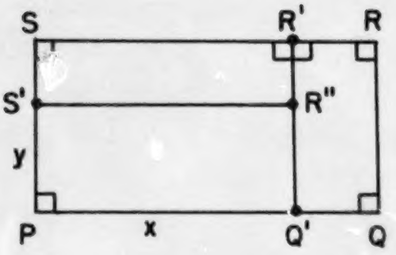
Restatement. Suppose a rectangle ABCD exists and x, y are given positive real numbers. Then there exists a rectangle PQRS such that $PQ > x$ and $PS > y$.



Proof. By the theorem we have a rectangle ABYZ with $AZ > x$. By placing successive congruent replicas of ABYZ "on top" of each other starting with ABYZ, we eventually get a rectangle AA'Z'Z with $AA' > y$ and $AZ > x$.

Theorem 2. If one particular rectangle exists then a rectangle exists with two adjacent sides of preassignedllengths x , y .

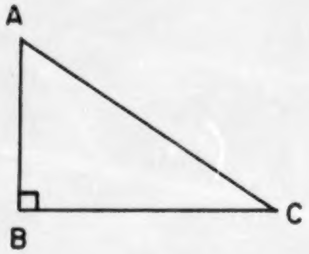
Proof. Our method is that of a tailor: By the last corollary we get a rectangle PQRS such that $PQ > x$ and $PS > y$; then we cut it down to fit.

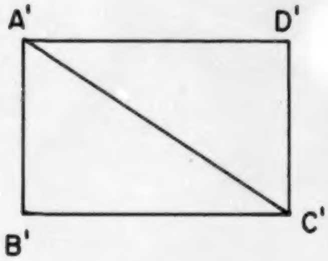


There is a point Q' in $\overline{PQ}$ such that PQ' = x . Drop a perpendicular from Q' to line $\overleftrightarrow{RS}$ with foot R' . We show PQ'R'S is a rectangle. It certainly has right angles at P, S, R'. We show $\angle$PQ'R' also is a right angle. Suppose $m\angle PQ'R' > 90$. Then the sum of the angle measures of quadrilateral PQ'R'S is greater than 360 contrary to the corollary of Legendre's Theorem (Part II). Suppose $m\angle PQ'R' < 90$. Then $m\angle QQ'R' > 90$ and quadrilateral QQ'R'R has an angle measure sum greater than 360. Thus the only possibility is $m\angle PQ'R' = 90$, and PQ'R'S is a rectangle.

In the same way there is a point S' in $\overline{PS}$ such that PS' = y . Drop a perpendicular from S' to line $\overleftrightarrow{Q'R'}$ with foot R'' . Then as above PQ'R''S' is a rectangle, and it has sides $\overline{PQ'}$ and $\overline{PS'}$ of lengths x and y .

Theorem 3. If one particular rectangle exists then every right triangle has an angle measure sum of 180.



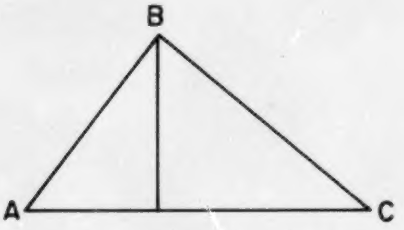


Proof. Our procedure is to show: (1) any right triangle is a replica of a triangle formed by the splitting of a rectangle by a diagonal, and (2) the latter type of triangle must have an angle measure sum of 180. Let ΔABC be a right triangle with right angle at B. By Theorem 2 there exists a rectangle $A'B'C'D'$ with $A'B' = AB$ and $B'C' = BC$. Draw $\overline{A'C'}$. Then $\Delta ABC = \Delta A'B'C'$ and they have the same angle measure sum. Let p be the angle measure sum of $\Delta A'B'C'$ and q be that of $\Delta A'C'D'$. We have

$$p + q = 4 \cdot 90 = 360 . \tag{1}$$

We want to show $p = 180$. By Legendre's Theorem $p < 180$ or $p = 180$. Suppose $p < 180$. Then by (1) $q > 180$, contrary to Legendre's Theorem. So $p = 180$ must hold and the proof is complete.

Theorem 4. If one particular rectangle exists then every triangle has an angle measure sum of 180.



Proof. Any triangle ΔABC can be split into two right triangles. Each of these has angle measure sum 180 by Theorem 3. It easily follows that the same holds for ΔABC.

This is a rather striking result: The existence of one puny rectangle with microscopic sides inhabiting a remote portion of space guarantees that every conceivable triangle has an angle measure sum of 180. Since this is a typically Euclidean property we are tempted to say that if in a neutral geometry a rectangle exists, the geometry must be Euclidean. The statement is correct but not fully justified, since to characterize a neutral geometry as Euclidean we must know that it satisfies Euclid's parallel postulate. This can now be proved without trouble.

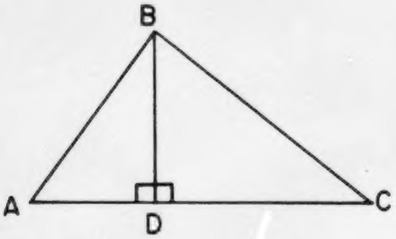
Theorem 5. If one particular rectangle exists then Euclid's parallel postulate holds.

Proof. Suppose a rectangle exists but Euclid's parallel postulate fails. Then there must exist a line L and a point P such that there are two lines through P parallel to L, since by Theorem 1 Corollary 3 of Part II there is at least one line parallel to a given line through an external point. Then by Theorem 2 of Part I there exists one triangle, at least, whose angle measure sum if less than 180. This contradicts Theorem 4. Consequently Euclid's parallel postulate must hold.

What we have justified is a remarkable equivalence theorem, namely: Euclid's parallel postulate is logically equivalent to the existence of a rectangle. That is taking either of these statements as a postulate we can deduce the other as a theorem, provided of course we assume the postulates for a neutral geometry.

An interesting condition equivalent to the existence of a rectangle is the existence of a triangle whose angle measure is 180:

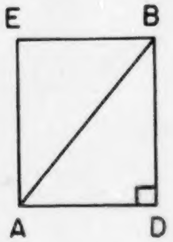
Theorem 6. If there exists one particular triangle with angle measure sum of 180, then there exists a rectangle.



Proof. Suppose ΔABC has angle measure sum 180. First we show there is a right triangle with angle measure sum 180. Split ΔABC into two right triangles, whose angle measure sums are say p and q. Then

$$p + q = 180 = 2 \cdot 90 = 360 .$$

We show $p = 180$. By Legendre's Theorem, $P \leq 180$. If $p < 180$ then $q > 180$ contrary to Legendre's Theorem. Thus there is a right triangle, say ΔABD , which has angle measure sum 180.



Now we put two such right triangles together to form a rectangle. Construct $\Delta AEB \cong \Delta BDA$ with E on the opposite side of line $\overleftrightarrow{AB}$ from D. Show $ADBE$ is a rectangle.

Corollary 1. If one particular triangle has angle measure sum 180 then every triangle has angle measure sum 180.

Proof. By Theorems 6, 4.

Corollary 2. If one particular triangle has angle measure sum 180 then Euclid's parallel postulate holds.

Proof. By Theorems 6, 5.

Corollary 3. If one particular triangle has an angle measure sum which is less than 180 then every triangle has an angle measure sum less than 180.

Proof. Suppose ΔABC has angle measure sum less than 180. Consider any triangle ΔPQR. By Legendre's Theorem its angle measure sum p must satisfy $p = 180$ or $p < 180$. Suppose $p = 180$. Then by Corollary 1, ΔABC has angle measure sum 180, contrary to hypothesis. Thus $p < 180$.

Comparing Corollaries 1 and 3 we observe an important fact. A neutral geometry is "homogeneous" in the sense that all of its triangles have an angle measure sum of 180 or they all have angle measure sums less than 180. The first type of neutral geometry is merely Euclidean geometry - the second type corresponds to the non-Euclidean geometry developed by Bolyai and Lobachevsky. This will be discussed in the next part.

Exercise 1.

Suppose there is only one line parallel to a particular line L through a particular point P. Prove that Euclid's parallel postulate holds.

Exercise 2.

Suppose there are two lines parallel to a particular line L through a particular point P Prove there are two lines parallel to each line through each external point

IV. Lobachevskian Geometry.

Now we introduce the non-Euclidean geometry of Bolyai and Lobachevsky as a formal theory based on its own postulates. We call the theory Lobachevskian geometry to signalize the lifetime of work which Lobachevsky devoted to the theory. To study Lobachevskian geometry we merely assume the postulates of Euclidean geometry but replace Euclid's parallel postulate by Lobachevsky's Parallel Postulate. If point P is not on line L there are at least two lines through P which are parallel to L . In other words we assume the postulates of neutral geometry (Postulates 1,...,15 of the text) and adjoin Lobachevsky's parallel postulate. Consequently the theorems which we have already derived are valid in Lobachevskian geometry. In fact, by putting together two earlier results we get the following important theorem.

Theorem 1. The angle measure sum of any triangle is less than 180.

Proof. By Theorem 2 of Part I there exists a triangle whose angle measure sum is less than 180. Hence the same is true of every triangle by Corollary 3 of Theorem 6, Part III.

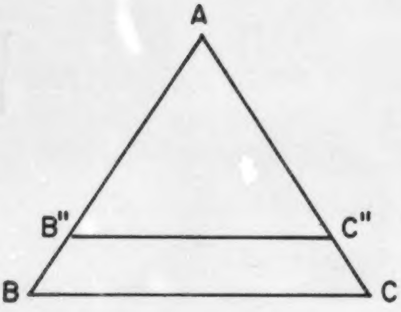
Corollary 1. The angle measure sum of any quadrilateral is less than 360.

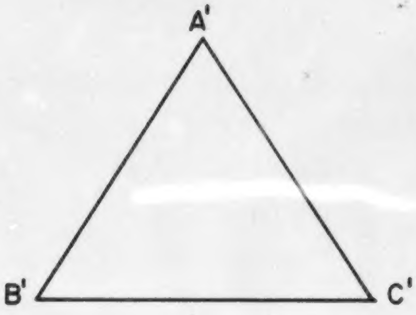
Proof. By the corollary to Legendre's Theorem (Part II Theorem 2) the only other possibility for the value is 360 - and this is ruled out by Theorem 1.

Corollary 2. There exist no rectangles.

Now we show that similar triangles can't exist in Lobachevskian geometry, except of course for the trivial case of congruent triangles.

Theorem 2. Two triangles are congruent if their corresponding angles have equal measures.





Proof. Suppose the theorem false. Then there exist ΔABC and $\Delta A'B'C'$ which are not congruent such that $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, $m\angle C = m\angle C'$. Since the triangles are not congruent $AB \neq A'B'$. Similarly $AC \neq A'C'$ and $BC \neq B'C'$. Consider the triples AB, AC, BC and $A'B', A'C', B'C'$. One of these triples must contain two numbers which are greater than the corresponding numbers of the other triple. Consequently it is not restrictive to suppose $AB > A'B'$ and $AC > A'C'$.

Then we can find B'' on $\overline{AB}$ such that $A'B' = AB''$ and C'' on $\overline{AC}$ such that $A'C' = AC''$. It follows that $\Delta AB''C'' \cong \Delta A'B'C'$ so that

$$m\angle AB''C'' = m\angle B' = m\angle B.$$

Hence $\angle BB''C''$ is supplementary to $\angle B$. Similarly $\angle CC''B''$ is supplementary to $\angle C$. Therefore quadrillateral $BB''C''C$ has an angle measure sum of 360. This contradicts Corollary 1 of the last theorem and our proof is complete.

We have here a striking contrast with Euclidean geometry. In view of Theorem 2, in Lobachevskian geometry there cannot be a theory of similar figures based on the usual definition. For if two triangles were similar, their corresponding angles would be equal and they would have to be congruent. In general two similar figures would be congruent and so have the same size. In a Lobachevskian world, pictures and statues would have to be life-size to avoid distortion.

Now let us consider the question of measurement of area. For the sake of simplicity we restrict ourselves to triangles. Clearly the Euclidean procedure of measuring area in terms of square units will not apply since squares don't exist in Lobachevskian geometry. To clarify the problem we ask what are the essential characteristics of area. As a minimum we require:

1. (1) The area of a triangle shall be a uniquely determined positive real number;
2. (2) Congruent triangles shall have equal areas;
3. (3) If a triangle T is split into two triangles T_1 and T_2 then the area of T shall be the sum of the areas of T_1 and T_2.

It is easy to verify that the familiar formula for the area of a triangle in Euclidean geometry satisfies these conditions.

There is a similar area formula (or area "function") in Lobachevskian geometry but it is most naturally expressed in terms of the angles of a triangle. To state it formally we introduce the

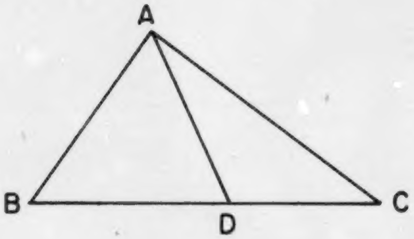
Definition. The defect (or deficiency) of ΔABC is $180-(m\angle A+m\angle B+m\angle C)$. Note that the defect of a triangle literally is the amount by which its angle measure sum falls short of 180.

The defect of a triangle has the essential properties of area:

Theorem 3. The defect of a triangle satisfies properties (1), (2), (3) above.

Proof. Clearly (1) is satisfied since the defect of a triangle is a definite positive number. Property (2) holds since congruent triangles have equal angle sums and so equal defects.

To establish (3) let ΔABC be given and let D be a point of $\overline{BC}$, so that ΔABC is split into ΔABD and ΔADC. The sum of the defects of the latter two triangles is



$$\begin{aligned} & 180 - (m\angle BAD+m\angle B+m\angle BDA)+180 - (m\angle CAD+m\angle C+m\angle CDA) \\ = {} & 180 - (m\angle BAD+m\angle CAD+m\angle B+m\angle C) \\ = {} & 180 - (m\angle BAC+m\angle B+m\angle C) \end{aligned}$$

which is the defect of ΔABC.

Are there other area functions besides the defect? It is easy to verify that if we multiply the defect by any positive constant k, we obtain an area function which satisfies properties (1), (2), (3). This is not as remarkable as it might seem, since the specific form of our definition of defect depends on our basic agreement to measure angles in terms of degrees. If we adopt a different unit for the measure of angles and define "defect" in the natural manner, we obtain a constant multiple of the defect as we defined it. To be specific, suppose we change the unit of angle measurement from degrees to minutes. This would entail two simple changes in the above theory: (a) each angle measure would have to be multiplied by 60; (b) the key number 180 would have to be replaced by 60 times 180. Thus the appropriate definition of "defect" would be 60 times the defect as we defined it.

Finally we note it can be proved that any area function satisfying (1), (2), (3) must be k times the defect (our definition) for some positive constant k. In view of this it is natural to define the area of a triangle to be its defect.

Query. Which of the properties (1), (2), (3) holds for the defect of a triangle in Euclidean geometry?

It is interesting to note that in Euclidean spherical geometry the sum of the angle measures of a triangle is greater than 180 and the area of a triangle is given by its "excess", that is its angle measure sum minus 180.

Exercise 1.

Given ΔABC with points D, E, F in $\overline{AB}$, $\overline{BC}$, $\overline{AC}$ respectively. Prove that the defect of ΔABC is the sum of the defects of the triangles ADF, BED, CFE, and DEF.

Exercise 2.

If points P, Q, R are inside ΔABC prove that ΔABC has a larger defect than ΔPQR.

We conclude this part by observing that the familiar Euclidean property - parallel lines are everywhere equidistant - fails in Lobachevskian geometry. In fact there are parallel lines of two types. If two parallel lines have a common perpendicular they diverge continuously on both sides of this perpendicular. If two parallel lines don't have a common perpendicular they are asymptotic - that is if a point on one recedes endlessly in the proper direction, its distance to the other will approach zero.

Conclusion.

In its further development Lobachevskian geometry is at least as complex as Euclidean geometry. There is a Lobachevskian solid geometry, a trigonometry and an analytic geometry - problems in mensuration of curves, surfaces and solids require the use of the calculus.

You may object that the structure is grounded on sand - that Lobachevskian geometry is inconsistent and eventually will yield contradictory theorems. This of course was the implicit belief that led mathematicians for 2,000 years to try to prove Euclid's parallel postulate. Actually we have no absolute test for the consistency of any of the familiar branches of mathematics. But it can be proved that the Euclidean and Lobachevskian geometries stand or fall together on the question of consistency. That is, if either is inconsistent, so is the other.

Once the ice had been broken by Bolyai and Lobachevsky's successful challenge to Euclid's parallel postulate, mathematicians were stimulated to set up other non-Euclidean geometries - that is, geometric theories which contradict one or more of Euclid's postulates, or approach geometry in an essentially different way. The best known of these was proposed in 1854 by the German mathematician Riemann (1826-1866). Riemann's theory contradicts Euclid's parallel postulate by assuming there are no parallel lines. This required the abandonment of other postulates of Euclid since we have proved the existence of parallel lines without assuming any parallel postulate (Part II, Theorem 1 Corollary 3). In Riemann's theory, in contrast to those of Euclid and Lobachevsky, a line has finite length. Actually there are two types of non-Euclidean geometry associated with Riemann's name, one called single elliptic geometry in which any two lines meet in just one point, and a second, double elliptic geometry, in which any two lines meet in two points The second type of geometry can be pictured in Euclidean space as the geometry of points and great circles on a sphere.

Riemann also introduced a radically different kind of geometric theory which builds up the properties of space in the large by studying the behavior of distance between points which are close together. This theory, called Riemannian geometry, is useful in applied mathematics and physics and is the mathematical basis of Einstein's general theory of relativity.

Bolyai and Lobachevsky have opened for us a door on a new and apparently limitless domain.